DYNAMIC OBSERVABILITY OF LATENT CONTAGION
SEQUENTIAL REVELATION AND OBSERVABLE SCREENING

J. VIDAL LLAURADÓ

Abstract. A latent Volterra contagion channel has no single visibility level. It can accumulate path-space evidence, improve oracle short-horizon prediction, and still be screened after conditioning on the target asset's own observable past. Dynamic observability is formulated as the residual predictive component left after latent drivers are replaced by observable Gaussian channels. In a bivariate Gaussian Volterra model this separates three experiments: finite-resolution path-space comparison, oracle prediction with latent drivers, and observable prediction in the target filtration.

In the smoothing regime $H_{XY} > H_Y$, the relative covariance operator generates a canonical finite-resolution Gaussian experiment. Exact likelihood, Kullback–Leibler, Hellinger, and Bayes-error formulas recover the path-detectability boundary $H_{XY} = H_Y + \frac{1}{4}$ as the critical rate at which evidence accumulates with resolution. At the oracle latent-driver level, short-horizon prediction has a different boundary, namely $H_{XY} = H_Y$. Rougher contagion improves prediction at leading order, smoother contagion is dynamically latent at leading order, and the boundary case places both channels on the same scale.

At the observable level, the oracle channel is compressed again. Passing from latent drivers to observed Gaussian channels sends dynamic information through a hidden transfer operator. In the rougher regime $0 < H_{XY} < H_Y < \frac{1}{2}$, a screening theorem is first proved under an h^α-stable boundary-cutoff hypothesis. The exact-correction closure verifies that hypothesis for the model classes considered here under the stated C^2 near-diagonal regularity assumptions. The endpoint-flat closure proves that the target's own past screens the leading oracle rough gain:

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = o(h^{H_{XY}}), \qquad \widetilde{G}_{X\to Y}(t,h) = o(h^{2H_{XY}}).$$

Pricing visibility, path-space detectability, and dynamic observability are therefore distinct visibility levels. The screened observable component is the inferential object passed to projected covariance-score inference.

1. Introduction

The threshold analysis of [1] separates pricing visibility from path-space detectability for smoothing cross-asset Volterra channels. That static separation leaves open which part of a latent channel survives once prediction is conditioned on observable histories. Dynamic observability is this residual predictive object.

Observable filtrations replace latent drivers by conditional laws generated by asset histories. Evidence accumulation, oracle short-horizon prediction, and target-past screened prediction are therefore different experiments. A channel may be detectable at finite resolution, may improve oracle prediction when latent drivers are resolved, and may still vanish from the leading gain after conditioning on the target asset's own past.

The bivariate Gaussian Volterra model keeps the covariance structure explicit enough to make these reductions exact. Asset histories enter through Gaussian channels; latent contagion enters through a cross Volterra kernel. The dynamic problem asks which component of the oracle signal remains in the observable filtration, rather than restating the smoothing threshold of [1] in time-indexed form.

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The probabilistic inputs are classical, but each one controls a different level. The finite-resolution comparison uses Gaussian experiment theory in the tradition of Le Cam [2], Le Cam and Yang [3], van der Vaart [4], and Ibragimov and Has'minskii [5], together with fractional LAN benchmarks such as Dahlhaus [6] and Brouste–Fukasawa [7]. The prediction problem uses fractional filtering results of Gripenberg and Norros [8], Kleptsyna and Le Breton [9], Nuzman and Poor [10], Cai, Chigansky, and Kleptsyna [11], and Afterman, Chigansky, Kleptsyna, and Marushkevych [12]. The observable reduction is a Volterra transfer problem, using the operator theory of Gripenberg, Londen, and Staffans [13], trace-ideal estimates in the sense of Simon [14], and the fractional covariance representation of Dzhaparidze and van Zanten [15].

The argument passes through three non-equivalent reductions. Theorem 2.6 converts the static path-space threshold into a finite-resolution Gaussian experiment. In the smoothing regime $H_{XY} > H_Y$, the relative covariance operator controls likelihood, Kullback–Leibler, Hellinger, and Bayes-error asymptotics. The formulas recover

$$H_{XY} = H_Y + \tfrac{1}{4}$$

as the critical evidence-accumulation boundary.

Theorem 3.12 separates static evidence from short-horizon prediction. At the oracle latent-driver level $H_Y + \frac{1}{4}$ no longer governs. The prediction boundary is

$$H_{XY} = H_Y.$$

Rougher contagion improves prediction at leading order, smoother contagion is dynamically latent at leading order, and the boundary case puts both channels on the same scale.

Observable screening is the final reduction. The target past is already contaminated by the contagion channel, so the source cannot be appended as an independent predictor. The screening theorem, Theorem 32.8, isolates the relation once h^α-stable boundary cutoffs are available. The exact-correction closure theorem, Theorem 33.52, verifies the cutoff hypothesis for the model classes considered here under the stated C^2 near-diagonal regularity assumptions. In the rougher regime $0 < H_{XY} < H_Y < \frac{1}{2}$, the endpoint-flat closure gives

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = o(h^{H_{XY}}) \qquad \text{and} \qquad \widetilde{G}_{X\to Y}(t,h) = o(h^{2H_{XY}}).$$

Pricing visibility, path-space detectability, and dynamic observability therefore need not coincide.

The screened component is an estimand before it is an estimator. Recent work on rough-volatility inference has sharpened what can be learned from high-frequency Gaussian or rough observations [7, 16, 17]. The object defined here precedes the choice of econometric procedure. A rough channel may exist while its observable content is smaller than both its pricing signature and its oracle predictive content.

The assumptions strengthen with the object. The static threshold model uses near-diagonal Volterra modulation at the C^1 level. Closed observable screening uses a C^2 near-diagonal expansion together with endpoint-flat cutoffs. Projected covariance-score inference requires that second-order screening structure plus the sampling and proxy-dominance conditions of the projected experiment. The hierarchy records assumption import across objects rather than an inconsistency between the threshold and screening statements.

Results by reduction. The finite-resolution Gaussian experiment comes first. It gives quantitative content to the path-detectability threshold of [1] and determines the rate at which statistical evidence accumulates with observation resolution.

The oracle prediction theorem separates prediction from path-space comparison. It identifies the dynamic threshold $H_{XY} = H_Y$ and proves that the short-horizon predictive problem is not the static detectability problem rewritten in time.

The observable screening block completes the reduction. It defines the inferential target used by feasible source-screened inference: the part of the oracle signal that remains after conditioning on the target channel and restricting information to observables. The finite-resolution experiment fixes the evidence scale. The oracle theorem gives the latent predictive benchmark. The screening theorem under h^α-stable boundary cutoffs isolates the screening relation, and the exact-correction theorem, Theorem 33.52, verifies the cutoff structure for the model classes treated here. The endpoint-flat closure fixes the observable object to be estimated.

Organization. Section 2 develops the finite-resolution Gaussian experiment and its information functionals in the smoothing regime. Sections 3 and 4 formulate the oracle and observable short-horizon prediction problems. Sections 5 through 18 build the exact Volterra reduction and the frozen fractional model. Sections 19 through 32 prove the screening theorem under h^{α}-stable boundary cutoffs. The final sections verify those cutoffs by exact correction, establish the closed observable theorem in the model classes treated here, and record the consequences in Section 34.

2. Finite-resolution information geometry

The first reduction turns the static smoothing threshold of [1] into a finite Gaussian experiment. The governing object is the positive compact relative covariance operator

$$\mathcal{R}_{Y,XY} := C_0^{-1/2}(C_1 - C_0)C_0^{-1/2}$$

and, in the smoothing regime, it satisfies

$$\tau_n(\mathcal{R}_{Y,XY}) \asymp n^{-2\delta}, \qquad \delta := H_{XY} - H_Y > 0,$$

where $\tau_n(\mathcal{R}_{Y,XY})$ denotes the nth eigenvalue in decreasing order. This eigenvalue order fixes the information scale of the finite-resolution experiment.

The same spectral profile generates the canonical finite-resolution Gaussian experiment. It also fixes the rate at which statistical evidence accumulates with the resolved dimension N or, equivalently, with an observation scale Δ such that $N_\Delta \asymp \Delta^{-1}$.

Definition 2.1 (Canonical N-mode experiment)**.** Let $\{\tau_n\}_{n\geq 1}$ be the eigenvalues of $\mathcal{R}_{Y,XY}$, repeated with multiplicity and ordered decreasingly. For each $N \in \mathbb{N}$, define the centred Gaussian laws

$$\nu_{0,N} := \bigotimes_{n=1}^{N} \mathcal{N}(0,1), \qquad \nu_{1,N} := \bigotimes_{n=1}^{N} \mathcal{N}(0,1+\tau_n)$$

on $\mathbb{R}^N$. The experiment

$$\mathcal{E}_N := (\mathbb{R}^N, \nu_{0,N}, \nu_{1,N})$$

is the *canonical N-mode experiment* associated with $\mathcal{R}_{Y,XY}$.

Remark 2.2 (Interpretation of the normal form)*.* Definition 2.1 is the spectral normal form associated with the relative covariance operator $\mathcal{R}_{Y,XY}$. In the Gaussian comparison problem, whitening and diagonalisation of the pair (C_0, C_1) reduce the finite-resolution comparison to the same eigenvalue profile $\{\tau_n\}$. All finite-resolution information quantities used below depend only on that profile. The experiment $\mathcal{E}_N$ is therefore the canonical object to analyse; see, for example, Le Cam [2], Le Cam and Yang [3, Ch. 2], van der Vaart [4], and Ibragimov and Has'minskii [5].

For later reference set

$$E_N := \sum_{n=1}^{N} \tau_n, \qquad I_N := \sum_{n=1}^{N} \tau_n^2.$$

The first quantity is a finite-resolution latent-energy diagnostic. The second is the quadratic information scale of the product Gaussian experiment. The evidence boundary below is governed by I_N, not by E_N, because likelihood, Kullback–Leibler, Hellinger, and Bayes-error asymptotics all reduce to the quadratic profile.

Proposition 2.3 (Likelihood ratio in normal form)**.** *For every $N \in \mathbb{N}$, the law $\nu_{1,N}$ is absolutely continuous with respect to $\nu_{0,N}$, with Radon–Nikodym derivative*

$$\frac{d\nu_{1,N}}{d\nu_{0,N}}(x) = \prod_{n=1}^{N}(1+\tau_n)^{-1/2} \exp\left(\frac{1}{2}\sum_{n=1}^{N}\frac{\tau_n}{1+\tau_n}x_n^2\right), \qquad x = (x_1,\ldots,x_N) \in \mathbb{R}^N. \tag{1}$$

Equivalently,

$$\log \frac{d\nu_{1,N}}{d\nu_{0,N}}(x) = -\frac{1}{2}\sum_{n=1}^{N} \log(1+\tau_n) + \frac{1}{2}\sum_{n=1}^{N} \frac{\tau_n}{1+\tau_n} x_n^2. \tag{2}$$

Proof. For one coordinate,

$$\frac{d\mathcal{N}(0,1+\tau_n)}{d\mathcal{N}(0,1)}(x_n) = (1+\tau_n)^{-1/2} \exp\left(-\frac{x_n^2}{2(1+\tau_n)} + \frac{x_n^2}{2}\right)$$

for $\tau_n \geq 0$. Since

$$-\frac{1}{2(1+\tau_n)} + \frac{1}{2} = \frac{\tau_n}{2(1+\tau_n)},$$

the one-dimensional density ratio is

$$(1+\tau_n)^{-1/2} \exp\left(\frac{\tau_n}{2(1+\tau_n)} x_n^2\right).$$

Independence of the coordinates under both product laws gives (1), and taking logarithms yields (2). □

Proposition 2.4 (Exact divergence functionals)**.** *For every $N \in \mathbb{N}$:*

(i) the two Kullback–Leibler divergences are

$$\mathrm{KL}(\nu_{1,N} \,\|\, \nu_{0,N}) = \frac{1}{2}\sum_{n=1}^{N} \Big(\tau_n - \log(1+\tau_n)\Big), \tag{3}$$

$$\mathrm{KL}(\nu_{0,N} \,\|\, \nu_{1,N}) = \frac{1}{2}\sum_{n=1}^{N} \left(\log(1+\tau_n) - \frac{\tau_n}{1+\tau_n}\right); \tag{4}$$

(ii) the Hellinger affinity is

$$\mathcal{A}_N := \int_{\mathbb{R}^N} \sqrt{\frac{d\nu_{1,N}}{dx}\frac{d\nu_{0,N}}{dx}}\, dx = \prod_{n=1}^{N} \frac{\sqrt{2}\,(1+\tau_n)^{1/4}}{(2+\tau_n)^{1/2}}; \tag{5}$$

(iii) if π_N^* denotes the optimal Bayes misclassification probability for equal priors, then

$$\frac{1}{2}\Big(1 - \sqrt{1-\mathcal{A}_N^2}\Big) \leq \pi_N^* \leq \frac{1}{2}\mathcal{A}_N. \tag{6}$$

Proof. For item (i), a centred Gaussian law $\mathcal{N}(0,\sigma^2)$ satisfies

$$\mathrm{KL}(\mathcal{N}(0,\sigma^2) \,\|\, \mathcal{N}(0,1)) = \frac{1}{2}(\sigma^2 - 1 - \log\sigma^2).$$

Applying this coordinatewise with $\sigma_n^2 = 1+\tau_n$ and using independence yields (3). Likewise,

$$\mathrm{KL}(\mathcal{N}(0,1) \,\|\, \mathcal{N}(0,\sigma^2)) = \frac{1}{2}\left(\log\sigma^2 + \frac{1}{\sigma^2} - 1\right),$$

which gives (4) after substituting $\sigma_n^2 = 1+\tau_n$ and simplifying

$$\frac{1}{1+\tau_n} - 1 = -\frac{\tau_n}{1+\tau_n}.$$

For item (ii), in one dimension the Hellinger affinity between $\mathcal{N}(0,\sigma_0^2)$ and $\mathcal{N}(0,\sigma_1^2)$ is

$$\sqrt{\frac{2\sigma_0\sigma_1}{\sigma_0^2+\sigma_1^2}}.$$

With $\sigma_0 = 1$ and $\sigma_1 = \sqrt{1+\tau_n}$ this becomes

$$\frac{\sqrt{2}\,(1+\tau_n)^{1/4}}{(2+\tau_n)^{1/2}}.$$

Multiplying over independent coordinates gives (5).

For item (iii), the optimal Bayes error with equal priors is

$$\pi_N^* = \frac{1}{2}\big(1 - \mathrm{TV}(\nu_{1,N}, \nu_{0,N})\big).$$

Also,

$$\mathrm{TV}(\nu_{1,N}, \nu_{0,N}) = \frac{1}{2}\int_{\mathbb{R}^N} \left|\sqrt{\frac{d\nu_{1,N}}{dx}} - \sqrt{\frac{d\nu_{0,N}}{dx}}\right| \left(\sqrt{\frac{d\nu_{1,N}}{dx}} + \sqrt{\frac{d\nu_{0,N}}{dx}}\right) dx.$$

By Cauchy–Schwarz,

$$\mathrm{TV}(\nu_{1,N}, \nu_{0,N}) \le \sqrt{1 - \mathcal{A}_N^2},$$

which gives the lower bound in (6). For the upper bound, using $\min(a,b) \le \sqrt{ab}$ for $a, b \ge 0$,

$$2\pi_N^* = \int_{\mathbb{R}^N} \min\left(\frac{d\nu_{1,N}}{dx}, \frac{d\nu_{0,N}}{dx}\right) dx \le \int_{\mathbb{R}^N} \sqrt{\frac{d\nu_{1,N}}{dx}\frac{d\nu_{0,N}}{dx}}\, dx = \mathcal{A}_N.$$

This proves (6). □

Lemma 2.5 (Quadratic control of the divergence profiles)**.** *There exist constants* $0 < c_1 \le c_2 < \infty$ *such that for every* $\tau \in [0,1]$*,*

$$c_1\tau^2 \le \tau - \log(1+\tau) \le c_2\tau^2, \tag{7}$$

$$c_1\tau^2 \le \log(1+\tau) - \frac{\tau}{1+\tau} \le c_2\tau^2, \tag{8}$$

$$c_1\tau^2 \le -\log\left(\frac{\sqrt{2}\,(1+\tau)^{1/4}}{(2+\tau)^{1/2}}\right) \le c_2\tau^2. \tag{9}$$

Proof. Define

$$\phi_1(\tau) := \tau - \log(1+\tau), \qquad \phi_0(\tau) := \log(1+\tau) - \frac{\tau}{1+\tau},$$

and

$$\psi(\tau) := -\log\left(\frac{\sqrt{2}\,(1+\tau)^{1/4}}{(2+\tau)^{1/2}}\right).$$

All three functions vanish at $\tau = 0$ and satisfy

$$\phi_1(\tau) = \frac{1}{2}\tau^2 + O(\tau^3), \qquad \phi_0(\tau) = \frac{1}{2}\tau^2 + O(\tau^3), \qquad \psi(\tau) = \frac{1}{16}\tau^2 + O(\tau^3)$$

as $\tau \downarrow 0$. Therefore the quotients

$$\frac{\phi_1(\tau)}{\tau^2}, \qquad \frac{\phi_0(\tau)}{\tau^2}, \qquad \frac{\psi(\tau)}{\tau^2}$$

extend continuously to $\tau = 0$ with positive finite limits. By continuity on the compact interval $[0,1]$, each quotient is bounded above and below by positive constants, which proves (7)–(9). □

Theorem 2.6 (Sharp information accumulation at resolution N)**.** *Assume there exist* $n_0 \in \mathbb{N}$*,* $\delta > 0$*, and constants* $0 < c_- \le c_+ < \infty$ *such that*

$$c_- n^{-2\delta} \le \tau_n \le c_+ n^{-2\delta}, \qquad n \ge n_0. \tag{10}$$

Then there exist constants $0 < C_1 \le C_2 < \infty$ *such that for all* $N \ge n_0$*:*

$$C_1 \sum_{n=1}^{N} n^{-4\delta} \le \mathrm{KL}(\nu_{1,N} \,\|\, \nu_{0,N}) \le C_2 \sum_{n=1}^{N} n^{-4\delta}, \tag{11}$$

$$C_1 \sum_{n=1}^{N} n^{-4\delta} \le \mathrm{KL}(\nu_{0,N} \,\|\, \nu_{1,N}) \le C_2 \sum_{n=1}^{N} n^{-4\delta}, \tag{12}$$

$$C_1 \sum_{n=1}^{N} n^{-4\delta} \le -\log \mathcal{A}_N \le C_2 \sum_{n=1}^{N} n^{-4\delta}. \tag{13}$$

Consequently, as $N \to \infty$,

$$\mathrm{KL}(\nu_{1,N} \,\|\, \nu_{0,N}),\ \mathrm{KL}(\nu_{0,N} \,\|\, \nu_{1,N}),\ -\log \mathcal{A}_N \asymp \begin{cases} N^{1-4\delta}, & 0 < \delta < \frac{1}{4}, \\ \log N, & \delta = \frac{1}{4}, \\ 1, & \delta > \frac{1}{4}. \end{cases} \tag{14}$$

Proof. Because $\tau_n \to 0$, choose $n_1 \geq n_0$ such that $\tau_n \leq 1$ for every $n \geq n_1$. Applying Lemma 2.5 termwise in the exact formulas (3)–(5), there exist constants $0 < C_1' \leq C_2' < \infty$ such that, for every $N \geq n_1$,

$$C_1' \sum_{n=n_1}^{N} \tau_n^2 \leq \mathrm{KL}(\nu_{1,N} \,\|\, \nu_{0,N}) \leq C_2' \sum_{n=n_1}^{N} \tau_n^2,$$
$$C_1' \sum_{n=n_1}^{N} \tau_n^2 \leq \mathrm{KL}(\nu_{0,N} \,\|\, \nu_{1,N}) \leq C_2' \sum_{n=n_1}^{N} \tau_n^2,$$
$$C_1' \sum_{n=n_1}^{N} \tau_n^2 \leq -\log \mathcal{A}_N \leq C_2' \sum_{n=n_1}^{N} \tau_n^2.$$

The eigenvalue bounds give

$$c_-^2 \sum_{n=n_1}^{N} n^{-4\delta} \leq \sum_{n=n_1}^{N} \tau_n^2 \leq c_+^2 \sum_{n=n_1}^{N} n^{-4\delta}.$$

The finite prefix $\{1, \ldots, n_1 - 1\}$ changes both the information functionals and the comparison series only by finite constants. Adjusting the constants on the finite set $n_0 \leq N < n_1$, and then on the finite prefix for $N \geq n_1$, gives (11)–(13) for every $N \geq n_0$.

The remaining step is the integral-test asymptotic

$$\sum_{n=1}^{N} n^{-4\delta} \asymp \begin{cases} N^{1-4\delta}, & 0 < \delta < \frac{1}{4}, \\ \log N, & \delta = \frac{1}{4}, \\ 1, & \delta > \frac{1}{4}. \end{cases}$$

Substitution into (11)–(13) gives (14). □

Corollary 2.7 (Observation-scale formulation)**.** *Let $\Delta \in (0,1)$ be an observation scale and set*

$$N_\Delta := \lfloor \Delta^{-1} \rfloor.$$

Under the assumptions of Theorem 2.6,

$$\mathrm{KL}(\nu_{1,N_\Delta} \,\|\, \nu_{0,N_\Delta}),\ \mathrm{KL}(\nu_{0,N_\Delta} \,\|\, \nu_{1,N_\Delta}),\ -\log \mathcal{A}_{N_\Delta} \asymp \begin{cases} \Delta^{-(1-4\delta)}, & 0 < \delta < \frac{1}{4}, \\ \log(\Delta^{-1}), & \delta = \frac{1}{4}, \\ 1, & \delta > \frac{1}{4}, \end{cases} \tag{15}$$

as $\Delta \downarrow 0$.

Proof. The relation $N_\Delta \asymp \Delta^{-1}$ transfers the three cases in (14) to the observation scale. If $0 < \delta < \frac{1}{4}$, then $N_\Delta^{1-4\delta} \asymp \Delta^{-(1-4\delta)}$. If $\delta = \frac{1}{4}$, then $\log N_\Delta \asymp \log(\Delta^{-1})$. If $\delta > \frac{1}{4}$, the rate remains of order one. This gives (15). □

Corollary 2.8 (Finite-resolution discrimination threshold)**.** *Under the assumptions of Theorem 2.6, the optimal Bayes error π_N^* satisfies:*

(i) if $0 < \delta \leq \frac{1}{4}$, then $\mathcal{A}_N \to 0$ and therefore

$$\pi_N^* \to 0 \qquad \text{as } N \to \infty;$$

(ii) if $\delta > \frac{1}{4}$, then $\mathcal{A}_N \to \mathcal{A}_\infty$ for some $\mathcal{A}_\infty \in (0,1]$, and

$$\liminf_{N\to\infty} \pi_N^* \geq \frac{1}{2}\Big(1 - \sqrt{1 - \mathcal{A}_\infty^2}\Big) > 0.$$

Proof. If $0 < \delta \le \frac{1}{4}$, then (13) implies $-\log \mathcal{A}_N \to \infty$, hence $\mathcal{A}_N \to 0$. The upper bound in (6) then yields $\pi_N^* \to 0$.

If $\delta > \frac{1}{4}$, then (13) shows that $-\log \mathcal{A}_N$ remains bounded as $N \to \infty$. Each Hellinger factor belongs to $0, 1]$, so $\mathcal{A}_N$ is decreasing. The boundedness of $-\log \mathcal{A}_N$ keeps the limit strictly positive, and therefore $\mathcal{A}_N \to \mathcal{A}_\infty \in (0, 1]$. The lower bound in (6) therefore gives

$$\liminf_{N\to\infty} \pi_N^* \ge \frac{1}{2}\Big(1 - \sqrt{1 - \mathcal{A}_\infty^2}\Big).$$

Because (10) gives $\tau_n > 0$ for every $n \ge n_0$, the Hellinger factor at n_0 is strictly smaller than 1. Hence $\mathcal{A}_N \le \mathcal{A}_{n_0} < 1$ for every $N \ge n_0$, and therefore $\mathcal{A}_\infty \le \mathcal{A}_{n_0} < 1$. Since the boundedness of $-\log \mathcal{A}_N$ gives $\mathcal{A}_\infty > 0$, the right-hand side is strictly positive. □

3. Short-horizon sequential prediction

The second reduction is predictive. The bivariate Volterra state already contains a short-horizon boundary before pricing or observable screening enters. The competition between the idiosyncratic exponent H_Y and the contagion exponent H_{XY} appears in prediction risk itself. For the prediction and filtering background used here, see in particular Gripenberg and Norros [8], Kleptsyna and Le Breton [9], Nuzman and Poor [10], and Afterman, Chigansky, Kleptsyna, and Marushkevych [12].

Assumption 3.1 (Bivariate Volterra volatility factor)**.** Fix $T > 0$. Let $W^\perp$ and W^X be independent standard Brownian motions. For $i \in \{Y, XY\}$ let $H_i \in (0, \frac{1}{2})$ and let L_i be defined on

$$\Delta_T := \{(t, s) \in [0, T]^2 : 0 \le s \le t \le T\},$$

with the following properties:

(i) L_i extends to a C^1 function on a neighbourhood of Δ_T;
(ii) L_i and its first partial derivatives are bounded on Δ_T;
(iii) $L_i(t, t) > 0$ for every $t \in [0, T]$.

Define the kernels

$$K_i(t, s) := (t - s)^{H_i - \frac{1}{2}} L_i(t, s) \mathbf{1}_{\{0 \le s < t \le T\}}, \qquad i \in \{Y, XY\}.$$

Let $\nu_Y > 0$ and $\eta \ge 0$, and define the Gaussian volatility factor

$$\mathcal{Y}_t := \nu_Y \int_0^t K_Y(t, s)\, dW_s^\perp \;+\; \eta \int_0^t K_{XY}(t, s)\, dW_s^X, \qquad 0 \le t \le T. \tag{16}$$

Definition 3.2 (Oracle intrinsic information filtrations)**.** For $0 \le t \le T$, define

$$\mathcal{G}_t^Y := \sigma(W_s^\perp : 0 \le s \le t), \qquad \mathcal{G}_t^{X,Y} := \sigma(W_s^\perp, W_s^X : 0 \le s \le t).$$

For $0 \le t < t + h \le T$, set

$$\Delta_h \mathcal{Y}_t := \mathcal{Y}_{t+h} - \mathcal{Y}_t.$$

The associated mean-square prediction risks are

$$R_Y(t, h) := \mathbb{E}\Big[\big(\Delta_h \mathcal{Y}_t - \mathbb{E}[\Delta_h \mathcal{Y}_t \mid \mathcal{G}_t^Y]\big)^2\Big], \tag{17}$$

$$R_{XY}(t, h) := \mathbb{E}\Big[\big(\Delta_h \mathcal{Y}_t - \mathbb{E}[\Delta_h \mathcal{Y}_t \mid \mathcal{G}_t^{X,Y}]\big)^2\Big]. \tag{18}$$

We also write

$$G_{X\to Y}(t, h) := R_Y(t, h) - R_{XY}(t, h) \tag{19}$$

for the prediction gain generated by the contagion channel.

Remark 3.3 (Oracle character of the intrinsic filtration block)*.* The filtrations of Definition 3.2 separate the idiosyncratic latent driver from the enlarged latent filtration that also contains the contagion driver up to time t. They are *not* yet the sigma-fields generated by observable asset prices or realised-volatility proxies. The section identifies the intrinsic short-horizon prediction gain available when the cross-asset latent channel is resolved exactly. This is the benchmark for the observable-filtration problem.

Definition 3.4 (Universal short-horizon constants)**.** For $H \in (0, \frac{1}{2})$, define

$$a_H := \frac{1}{2H}, \tag{20}$$

and

$$b_H := \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du. \tag{21}$$

Also set

$$q_H := a_H + b_H. \tag{22}$$

Lemma 3.5 (Finiteness of the past-resolving constant)**.** *For every $H \in (0, \frac{1}{2})$ one has $b_H \in (0, \infty)$.*

Proof. Near $u = 0$, since $H - \frac{1}{2} < 0$,

$$\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 \asymp u^{2H-1},$$

and u^{2H-1} is integrable at 0 because $2H - 1 > -1$.

As $u \to \infty$, the mean value theorem gives

$$(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} = \left(H - \frac{1}{2}\right) \xi_u^{H-\frac{3}{2}}$$

for some $\xi_u \in (u, u+1)$, hence

$$\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 \lesssim u^{2H-3},$$

and u^{2H-3} is integrable at ∞ because $2H - 3 < -2$. The integrand is not identically zero, so $b_H > 0$. □

Proposition 3.6 (Exact oracle projection formulas for the prediction risks)**.** *Fix $0 \le t < t+h \le T$. Then*

$$R_{XY}(t,h) = \nu_Y^2 \int_t^{t+h} K_Y(t+h,s)^2\, ds + \eta^2 \int_t^{t+h} K_{XY}(t+h,s)^2\, ds, \tag{23}$$

$$R_Y(t,h) = \nu_Y^2 \int_t^{t+h} K_Y(t+h,s)^2\, ds + \eta^2\, \mathrm{Var}(\Delta_h Z_t), \tag{24}$$

$$G_{X\to Y}(t,h) = \eta^2 \int_0^t \left(K_{XY}(t+h,s) - K_{XY}(t,s)\right)^2 ds, \tag{25}$$

where

$$Z_t := \int_0^t K_{XY}(t,s)\, dW_s^X.$$

Equivalently,

$$\mathrm{Var}(\Delta_h Z_t) = \int_0^t \left(K_{XY}(t+h,s) - K_{XY}(t,s)\right)^2 ds + \int_t^{t+h} K_{XY}(t+h,s)^2\, ds. \tag{26}$$

Proof. Define

$$V_t := \int_0^t K_Y(t,s)\, dW_s^\perp, \qquad Z_t := \int_0^t K_{XY}(t,s)\, dW_s^X.$$

By (16),

$$\Delta_h \mathcal{Y}_t = \nu_Y \Delta_h V_t + \eta \Delta_h Z_t,$$

where

$$\Delta_h V_t = \int_0^t \left(K_Y(t+h,s) - K_Y(t,s)\right) dW_s^\perp + \int_t^{t+h} K_Y(t+h,s)\, dW_s^\perp,$$

and

$$\Delta_h Z_t = \int_0^t \left(K_{XY}(t+h,s) - K_{XY}(t,s)\right) dW_s^X + \int_t^{t+h} K_{XY}(t+h,s)\, dW_s^X.$$

Conditioning on $\mathcal{G}_t^{X,Y}$, the integrals over $[0,t]$ are measurable, while the integrals over $(t,t+h]$ are independent centred Gaussians. Therefore

$$\Delta_h \mathcal{Y}_t - \mathbb{E}[\Delta_h \mathcal{Y}_t \mid \mathcal{G}_t^{X,Y}] = \nu_Y \int_t^{t+h} K_Y(t+h,s)\, dW_s^{\perp} + \eta \int_t^{t+h} K_{XY}(t+h,s)\, dW_s^X.$$

Applying Itô isometry and independence yields (23).

Conditioning instead on $\mathcal{G}_t^{Y}$, the term $\Delta_h Z_t$ is a centred functional of W^X and is independent of the sigma-field generated by $W^{\perp}$ up to time t, so

$$\Delta_h \mathcal{Y}_t - \mathbb{E}[\Delta_h \mathcal{Y}_t \mid \mathcal{G}_t^{Y}] = \nu_Y \int_t^{t+h} K_Y(t+h,s)\, dW_s^{\perp} + \eta\, \Delta_h Z_t.$$

The two displayed summands are independent, so Itô isometry gives

$$R_Y(t,h) = \nu_Y^2 \int_t^{t+h} K_Y(t+h,s)^2\, ds + \eta^2 \operatorname{Var}(\Delta_h Z_t),$$

which is (24).

Finally, subtracting (23) from (24) and using the decomposition of $\Delta_h Z_t$ into the past-resolved and future-innovation pieces gives (25) and (26). □

Proposition 3.7 (Innovation variance asymptotics). *Let $H \in (0, \frac{1}{2})$ and let L satisfy the regularity assumptions of Assumption 3.1. Define*

$$K_H(t,s) := (t-s)^{H-\frac{1}{2}} L(t,s) \mathbf{1}_{\{0 \le s < t \le T\}}.$$

Then for every fixed $t \in [0,T)$,

$$\int_t^{t+h} K_H(t+h,s)^2\, ds = L(t,t)^2 a_H h^{2H} + o(h^{2H}), \qquad h \downarrow 0. \tag{27}$$

Proof. By the change of variables $s = t+h-hu$,

$$\int_t^{t+h} K_H(t+h,s)^2\, ds = h^{2H} \int_0^1 u^{2H-1} L(t+h, t+h-hu)^2\, du.$$

Since L is continuous on a neighbourhood of the diagonal,

$$L(t+h, t+h-hu) \to L(t,t) \qquad \text{for every } u \in [0,1]$$

as $h \downarrow 0$, and the integrand is dominated by Cu^{2H-1} for some constant $C < \infty$. Because u^{2H-1} is integrable on $(0,1)$, dominated convergence yields

$$h^{-2H} \int_t^{t+h} K_H(t+h,s)^2\, ds \longrightarrow L(t,t)^2 \int_0^1 u^{2H-1}\, du = L(t,t)^2 a_H.$$

This proves (27). □

Proposition 3.8 (Past-resolved variance asymptotics). *Let $H \in (0, \frac{1}{2})$ and let L satisfy the regularity assumptions of Assumption 3.1. Define K_H as in Proposition 3.7. Then for every fixed $t \in (0,T)$,*

$$\int_0^t \big(K_H(t+h,s) - K_H(t,s)\big)^2\, ds = L(t,t)^2 b_H h^{2H} + o(h^{2H}), \qquad h \downarrow 0. \tag{28}$$

Proof. Write $\alpha := H - \frac{1}{2} \in (-\frac{1}{2}, 0)$ and change variables $u = t-s$. Then

$$I_h := \int_0^t \big(K_H(t+h,s) - K_H(t,s)\big)^2\, ds = \int_0^t \Big((u+h)^{\alpha} L(t+h, t-u) - u^{\alpha} L(t, t-u)\Big)^2\, du.$$

Split $I_h = I_h^{\text{near}} + I_h^{\text{far}}$, where

$$I_h^{\text{near}} := \int_0^{h^{1/2}} (\cdots)^2\, du, \qquad I_h^{\text{far}} := \int_{h^{1/2}}^t (\cdots)^2\, du.$$

Near-diagonal part. Since L is C^1, there exists $C < \infty$ such that for all small h and all $u \in [0, h^{1/2}]$,

$$|L(t+h, t-u) - L(t,t)| + |L(t, t-u) - L(t,t)| \le C(h+u) \le 2Ch^{1/2}.$$

Hence

$$\begin{aligned}(u+h)^\alpha L(t+h,t-u) &- u^\alpha L(t,t-u)\\ &= L(t,t)\big((u+h)^\alpha - u^\alpha\big) + r_h(u),\end{aligned}$$

with

$$|r_h(u)| \le C h^{1/2}\big((u+h)^\alpha + u^\alpha\big).$$

Therefore

$$I_h^{\text{near}} = L(t,t)^2 \int_0^{h^{1/2}} \big((u+h)^\alpha - u^\alpha\big)^2 \, du + o(h^{2H}),$$

because

$$\int_0^{h^{1/2}} r_h(u)^2 \, du \lesssim h \int_0^{h^{1/2}} \big((u+h)^{2H-1} + u^{2H-1}\big) \, du = O(h^{1+H}) = o(h^{2H}).$$

The main model term is $O(h^{2H})$ by Lemma 3.5, so the corresponding cross term is $o(h^{2H})$ by Cauchy–Schwarz.

Now substitute $u = hv$ in the main term:

$$\int_0^{h^{1/2}} \big((u+h)^\alpha - u^\alpha\big)^2 \, du = h^{2H} \int_0^{h^{-1/2}} \big((1+v)^\alpha - v^\alpha\big)^2 \, dv.$$

By Lemma 3.5, the integrand is integrable on $(0,\infty)$, so

$$h^{-2H} I_h^{\text{near}} \longrightarrow L(t,t)^2 b_H.$$

Far-from-diagonal part. For $u \ge h^{1/2}$, define

$$F_h(u) := (u+h)^\alpha L(t+h, t-u).$$

The mean value theorem in h gives

$$|F_h(u) - F_0(u)| \le h \sup_{0 \le r \le h} \left|\partial_r\big((u+r)^\alpha L(t+r,t-u)\big)\right|.$$

Since $u \ge h^{1/2}$ and $L, \partial_1 L$ are bounded,

$$|F_h(u) - F_0(u)| \le Ch(u^{\alpha-1} + u^\alpha)$$

for some constant C. Consequently,

$$\begin{aligned}I_h^{\text{far}} &\lesssim h^2 \int_{h^{1/2}}^t (u^{2\alpha-2} + u^{2\alpha}) \, du\\ &= O\Big(h^2 \int_{h^{1/2}}^t u^{2H-3} \, du\Big) + O(h^2)\\ &= O(h^{1+H}) + O(h^2) = o(h^{2H}),\end{aligned}$$

because $H < \frac{1}{2}$.

Combining the near and far parts proves (28). □

Corollary 3.9 (Short-horizon increment asymptotics)**.** *Under the assumptions of Proposition 3.8,*

$$\operatorname{Var}(X_{t+h} - X_t) = L(t,t)^2 q_H h^{2H} + o(h^{2H}), \qquad h \downarrow 0, \tag{29}$$

for the Gaussian Volterra process

$$X_t := \int_0^t K_H(t,s) \, dW_s.$$

Proof. The increment decomposes as

$$X_{t+h} - X_t = \int_0^t \big(K_H(t+h,s) - K_H(t,s)\big) \, dW_s + \int_t^{t+h} K_H(t+h,s) \, dW_s.$$

The two integrals are independent centred Gaussian variables because they are supported on disjoint Brownian time intervals. Itô isometry therefore gives the sum of the two variances, and Propositions 3.7 and 3.8 give the constants a_H and b_H. Since $q_H = a_H + b_H$, (29) follows. □

Lemma 3.10 (Off-diagonal pointwise increment bound)**.** *Let $H \in (0, \frac{1}{2})$ and let L satisfy the regularity assumptions of Assumption 3.1. Define K_H as in Proposition 3.7. Then for every fixed $t \in (0,T)$ there exist constants $C < \infty$ and $h_0 \in (0,t]$ such that for every $0 < h \leq h_0$ and every $s \in [0, t-h]$,*

$$|K_H(t+h,s) - K_H(t,s)| \leq Ch\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big). \tag{30}$$

Proof. Write $\alpha := H - \frac{1}{2} \in (-\frac{1}{2}, 0)$ and define

$$F(r,s) := (r-s)^{\alpha} L(r,s), \qquad 0 \leq s < r \leq T.$$

Fix $t \in (0,T)$ and choose

$$h_0 \in (0, \min\{t, T-t, 1\}] \,.$$

By the regularity assumptions on L, there exists $C < \infty$ such that for all $0 \leq s \leq t$ and all $r \in [t, t+h_0]$ with $r > s$,

$$|L(r,s)| + |\partial_1 L(r,s)| \leq C.$$

For $s \in [0, t-h]$, the mean value theorem gives

$$K_H(t+h,s) - K_H(t,s) = F(t+h,s) - F(t,s) = h\, \partial_1 F(\xi, s)$$

for some $\xi \in (t, t+h)$. Because $s \leq t-h < \xi$, one has

$$\partial_1 F(\xi,s) = \alpha(\xi-s)^{\alpha-1} L(\xi,s) + (\xi-s)^{\alpha} \partial_1 L(\xi,s),$$

and therefore

$$|\partial_1 F(\xi,s)| \leq C\big((\xi-s)^{H-\frac{3}{2}} + (\xi-s)^{H-\frac{1}{2}}\big).$$

Since $H - \frac{3}{2} < 0$ and $H - \frac{1}{2} < 0$, while $\xi - s \geq t - s$, we may absorb the resulting comparison into the constant and obtain

$$|\partial_1 F(\xi,s)| \leq C\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big).$$

Multiplying by h proves (30). □

Lemma 3.11 (L^2-continuity of Volterra channels)**.** *Under the assumptions of Proposition 3.8, the Gaussian Volterra process*

$$X_t := \int_0^t K_H(t,s)\, dW_s, \qquad 0 \leq t \leq T,$$

is continuous from $[0,T]$ into $L^2(\Omega)$.

Proof. For $t \in [0,T]$, define

$$k_t(s) := K_H(t,s)\mathbf{1}_{\{0 \leq s \leq t\}}, \qquad 0 \leq s \leq T.$$

Then $X_t = \int_0^T k_t(s)\, dW_s$, so by Itô isometry

$$\|X_u - X_t\|^2_{L^2(\Omega)} = \|k_u - k_t\|^2_{L^2([0,T])}.$$

It therefore suffices to show that $t \mapsto k_t$ is continuous in $L^2([0,T])$.

If $t = 0$, Proposition 3.7 gives

$$\|k_u\|^2_{L^2([0,T])} = \int_0^u K_H(u,s)^2\, ds = L(0,0)^2 a_H u^{2H} + o(u^{2H}) \longrightarrow 0$$

as $u \downarrow 0$.

Fix now $t \in (0,T]$ and let $u \to t$. Choose $\delta \in (0,t)$ so small that $u \in [t-\delta, t+\delta] \cap [0,T]$ for all u sufficiently close to t. Then

$$\begin{aligned}\|k_u - k_t\|^2_{L^2([0,T])} = &\int_0^{t-\delta} \big(K_H(u,s) - K_H(t,s)\big)^2\, ds \\ &+ \int_{t-\delta}^{\max\{u,t\}} \big(k_u(s) - k_t(s)\big)^2\, ds.\end{aligned}$$

On $[0,t-\delta]$, the map $(r,s)\mapsto K_H(r,s)$ is continuous and bounded on the compact set $[t-\delta,t+\delta]\times[0,t-\delta]$, so dominated convergence yields

$$\int_0^{t-\delta}\big(K_H(u,s)-K_H(t,s)\big)^2\,ds\longrightarrow 0.$$

For the near-diagonal part, there exists $C<\infty$ such that for all $r\in[t-\delta,t+\delta]\cap[0,T]$ and all $s<r$,

$$|K_H(r,s)|\le C(r-s)^{H-\frac12}.$$

Hence

$$\begin{aligned}\int_{t-\delta}^{\max\{u,t\}}\big(k_u(s)-k_t(s)\big)^2\,ds&\le 2\int_{t-\delta}^{\max\{u,t\}}k_u(s)^2\,ds+2\int_{t-\delta}^{t}k_t(s)^2\,ds\\&\le C\int_{t-\delta}^{\max\{u,t\}}\big((u-s)_+^{2H-1}+(t-s)_+^{2H-1}\big)\,ds\\&\le C\delta^{2H}.\end{aligned}$$

Taking $\limsup_{u\to t}$ and then letting $\delta\downarrow 0$ proves $\|k_u-k_t\|_{L^2([0,T])}\to 0$. □

Theorem 3.12 (Oracle dynamic prediction threshold)**.** *Assume the bivariate Volterra model of Assumption 3.1. Fix $t\in(0,T)$. Then, as $h\downarrow 0$,*

$$R_{XY}(t,h)=\nu_Y^2L_Y(t,t)^2a_{H_Y}h^{2H_Y}+\eta^2L_{XY}(t,t)^2a_{H_{XY}}h^{2H_{XY}}+o\big(h^{2H_Y}+h^{2H_{XY}}\big),\tag{31}$$

$$G_{X\to Y}(t,h)=\eta^2L_{XY}(t,t)^2b_{H_{XY}}h^{2H_{XY}}+o(h^{2H_{XY}}),\tag{32}$$

$$R_Y(t,h)=\nu_Y^2L_Y(t,t)^2a_{H_Y}h^{2H_Y}+\eta^2L_{XY}(t,t)^2q_{H_{XY}}h^{2H_{XY}}+o\big(h^{2H_Y}+h^{2H_{XY}}\big).\tag{33}$$

If $\eta>0$, then

$$\frac{G_{X\to Y}(t,h)}{R_Y(t,h)}\longrightarrow\begin{cases}\dfrac{b_{H_{XY}}}{q_{H_{XY}}}, & H_{XY}<H_Y,\\[2ex]\dfrac{\eta^2L_{XY}(t,t)^2b_{H_Y}}{\nu_Y^2L_Y(t,t)^2a_{H_Y}+\eta^2L_{XY}(t,t)^2q_{H_Y}}, & H_{XY}=H_Y,\\[2ex] 0, & H_{XY}>H_Y.\end{cases}\tag{34}$$

If $\eta=0$, then $G_{X\to Y}(t,h)\equiv 0$. Thus, for a nonzero contagion channel, the dynamic prediction threshold is

$$H_{XY}=H_Y.$$

Below it the contagion channel improves prediction at leading order, at it the gain is of leading order but mixed with the idiosyncratic term, and above it the gain is asymptotically negligible at leading order.

Proof. The exact formulas (23)–(25), together with Propositions 3.7 and 3.8, give the asymptotic expansions (31)–(33). If $\eta=0$, equation (25) gives $G_{X\to Y}(t,h)\equiv 0$, so the null-channel statement follows. Hence assume $\eta>0$ for the gain-ratio limits.

If $H_{XY}<H_Y$, then $h^{2H_{XY}}$ decays more slowly than h^{2H_Y}, so the contagion contribution is the leading term in $R_Y(t,h)$ and

$$R_Y(t,h)=\eta^2L_{XY}(t,t)^2q_{H_{XY}}h^{2H_{XY}}(1+o(1)).$$

Combining this with (32) yields the first limit in (34).

If $H_{XY}=H_Y$, both terms are of the same order and the second limit in (34) follows by dividing numerator and denominator by h^{2H_Y} and retaining the two coefficients.

If $H_{XY}>H_Y$, then

$$R_Y(t,h)=\nu_Y^2L_Y(t,t)^2a_{H_Y}h^{2H_Y}(1+o(1)),$$

while $G_{X\to Y}(t,h)=O(h^{2H_{XY}})$. Therefore

$$\frac{G_{X\to Y}(t,h)}{R_Y(t,h)}=O\big(h^{2(H_{XY}-H_Y)}\big)\longrightarrow 0.$$

This proves (34). □

Remark 3.13 (Interpretation of the oracle prediction block). Theorem 3.12 is the time-dynamic analogue of the pricing threshold of [1] at the *oracle latent-driver level*. It proves that, at short horizons, the predictive value of the contagion channel is governed by the same exponent comparison H_{XY} versus H_Y. Rougher contagion is dynamically visible at leading order, smoother contagion is dynamically latent at leading order, and the boundary case places both channels on the same scale.

Remark 3.14 (Size of the leading-order gain). Since $a_H > 0$ and $b_H > 0$, one has

$$0 < \frac{b_H}{q_H} < 1, \qquad q_H = a_H + b_H.$$

Thus in the rougher-contagion regime $H_{XY} < H_Y$, the enlarged oracle filtration removes a nontrivial but not total fraction of the leading-order prediction risk: the future innovation on $(t, t+h]$ remains unpredictable even after the past contagion driver is resolved.

4. Observed-channel Gaussian reduction

Section 3 resolves the latent contagion driver directly. The observed-channel reduction changes the information set. The latent source driver is replaced by an observed Gaussian source channel, and the problem becomes the posterior-variance reduction induced by observing the target past alone.

Assumption 4.1 (Observed-channel identifiability)**.** Let $H_X \in (0, \frac{1}{2})$ and let L_X satisfy the same regularity conditions as in Assumption 3.1. Define

$$K_X^{\mathrm{obs}}(t,s) := (t-s)^{H_X - \frac{1}{2}} L_X(t,s) \mathbf{1}_{\{0 \le s < t \le T\}}$$

and the observed source channel

$$X_t^{\mathrm{obs}} := \int_0^t K_X^{\mathrm{obs}}(t,s)\, dW_s^X, \qquad 0 \le t \le T. \tag{35}$$

For each $t \in (0,T]$, let

$$\mathcal{K}_{X,t}^{\mathrm{obs}} : L^2([0,t]) \to L^2([0,t])$$

be the associated Volterra operator

$$(\mathcal{K}_{X,t}^{\mathrm{obs}} f)(r) := \int_0^r K_X^{\mathrm{obs}}(r,s) f(s)\, ds, \qquad 0 \le r \le t.$$

Also, for each $t \in (0,T]$, let

$$\mathcal{K}_{Y,t} : L^2([0,t]) \to L^2([0,t]), \qquad (\mathcal{K}_{Y,t} f)(r) := \int_0^r K_Y(r,s) f(s)\, ds.$$

Assume that both $\mathcal{K}_{X,t}^{\mathrm{obs}}$ and $\mathcal{K}_{Y,t}$ are injective for every $t \in (0,T]$.

Remark 4.2 (Interpretation of the identifiability hypotheses). Injectivity of $\mathcal{K}_{X,t}^{\mathrm{obs}}$ means that the observed source channel does not lose first-chaos information about the source driver on $[0,t]$, while injectivity of $\mathcal{K}_{Y,t}$ means that once the contagion component is removed from the target, the remaining idiosyncratic channel still determines the past Brownian driver $W^\perp$ on $[0,t]$. These are exactly the hypotheses needed to identify the enlarged observed filtration with the enlarged oracle filtration.

Lemma 4.3 (Closedness of measurable L^2 subspaces)**.** *Let $\mathcal{F}$ be a sigma-field on Ω. Then*

$$L^2(\mathcal{F}) := \Big\{ X \in L^2(\Omega) : X \text{ admits an } \mathcal{F}\text{-measurable version} \Big\}$$

is a closed linear subspace of $L^2(\Omega)$.

Proof. Linearity follows by taking measurable versions and forming finite linear combinations. To prove closedness, let $X_n \in L^2(\mathcal{F})$ and suppose that

$$X_n \to X \qquad \text{in } L^2(\Omega).$$

Choose $\mathcal{F}$-measurable versions $\widetilde{X}_n$ of X_n. The same convergence holds for $\widetilde{X}_n$, so a subsequence $\widetilde{X}_{n_k}$ converges to X almost surely. The almost-sure limit of $\mathcal{F}$-measurable random variables is $\mathcal{F}$-measurable after redefining on a null set. Thus X admits an $\mathcal{F}$-measurable version, and $X \in L^2(\mathcal{F})$. ☐

Lemma 4.4 (Gaussian first-chaos criterion)**.** *Let $\{G_\alpha\}_{\alpha\in A}$ and $\{H_\beta\}_{\beta\in B}$ be centred Gaussian families on the same probability space, and let*

$$\mathfrak{H}_G := \overline{\operatorname{span}}\{G_\alpha : \alpha \in A\}, \qquad \mathfrak{H}_H := \overline{\operatorname{span}}\{H_\beta : \beta \in B\}$$

be the corresponding closed subspaces of $L^2(\Omega)$. If $\mathfrak{H}_G = \mathfrak{H}_H$, then

$$\sigma(G_\alpha : \alpha \in A) = \sigma(H_\beta : \beta \in B) \qquad \text{mod } \mathbb{P}.$$

Proof. Every finite linear combination of the G_α is measurable with respect to $\sigma(G_\alpha : \alpha \in A)$. Lemma 4.3 therefore implies that the closed span $\mathfrak{H}_G$ is contained in

$$L^2\big(\sigma(G_\alpha : \alpha \in A)\big),$$

and hence

$$\sigma(\mathfrak{H}_G) \subseteq \sigma(G_\alpha : \alpha \in A).$$

Conversely, each G_α belongs to $\mathfrak{H}_G$, so

$$\sigma(G_\alpha : \alpha \in A) \subseteq \sigma(\mathfrak{H}_G).$$

Thus $\sigma(G_\alpha : \alpha \in A) = \sigma(\mathfrak{H}_G)$ modulo null sets. The same argument applies to the H_β-family. If $\mathfrak{H}_G = \mathfrak{H}_H$, the generated sigma-fields coincide modulo $\mathbb{P}$. ☐

Lemma 4.5 (L^2-continuous families and integral spans)**.** *Let $t > 0$ and let $G : [0,t] \to L^2(\Omega)$ be continuous. Then*

$$\overline{\operatorname{span}}^{L^2(\Omega)}\{G(r) : 0 \le r \le t\} = \overline{\left\{\int_0^t f(r)G(r)\, dr : f \in L^2([0,t])\right\}}^{L^2(\Omega)}.$$

Proof. Every Bochner integral $\int_0^t f(r)G(r)\, dr$ belongs to the closed linear span of $\{G(r) : 0 \le r \le t\}$, so only the reverse inclusion needs proof.

Fix $r_0 \in [0,t]$. Choose a sequence of intervals $I_n \subset [0,t]$ such that $r_0 \in I_n$, $|I_n| > 0$, and $|I_n| \downarrow 0$. Let

$$f_n(r) := \frac{\mathbf{1}_{I_n}(r)}{|I_n|}.$$

Then $f_n \in L^2([0,t])$ and

$$\int_0^t f_n(r)G(r)\, dr = \frac{1}{|I_n|}\int_{I_n} G(r)\, dr.$$

By Jensen's inequality and L^2-continuity of G at r_0,

$$\left\| \frac{1}{|I_n|}\int_{I_n} G(r)\, dr - G(r_0)\right\|^2_{L^2(\Omega)} \le \frac{1}{|I_n|}\int_{I_n} \|G(r) - G(r_0)\|^2_{L^2(\Omega)}\, dr \longrightarrow 0.$$

Hence each $G(r_0)$ lies in the closed linear span of the integral family, proving the claim. ☐

Proposition 4.6 (Single-driver Volterra filtration transfer)**.** *Fix $t \in (0,T]$, let B be a standard Brownian motion, and let*

$$\mathcal{K}_t : L^2([0,t]) \to L^2([0,t])$$

be a Volterra operator of the form

$$(\mathcal{K}_t f)(r) := \int_0^r K(r,s)f(s)\, ds, \qquad 0 \le r \le t.$$

Assume that $\mathcal{K}_t$ is injective, and define

$$U_r := \int_0^r K(r,s)\ dB_s, \qquad 0 \le r \le t.$$

Assume in addition that $r \mapsto U_r$ is continuous from $[0,t]$ into $L^2(\Omega)$. Then

$$\sigma(U_r : 0 \le r \le t) = \sigma(B_r : 0 \le r \le t) \qquad \textit{mod}\ \mathbb{P}. \tag{36}$$

Proof. Let

$$I_t^B(f) := \int_0^t f(s)\ dB_s, \qquad f \in L^2([0,t]).$$

By Itô isometry, I_t^B is an isometric isomorphism from $L^2([0,t])$ onto the first chaos generated by $\{B_r : 0 \le r \le t\}$.

For $f \in L^2([0,t])$, boundedness of $\mathcal{K}_t$ gives $\mathcal{K}_t^* f \in L^2([0,t])$. The square-integrable stochastic Fubini theorem therefore applies and gives

$$\begin{aligned}\int_0^t f(r) U_r\ dr &= \int_0^t f(r) \int_0^r K(r,s)\ dB_s\ dr \\ &= \int_0^t \left(\int_s^t f(r) K(r,s)\ dr \right) dB_s \\ &= I_t^B(\mathcal{K}_t^* f).\end{aligned}$$

Lemma 4.5 therefore gives

$$\overline{\operatorname{span}}^{L^2(\Omega)}\{U_r : 0 \le r \le t\} = I_t^B(\overline{\operatorname{Ran}(\mathcal{K}_t^*)}).$$

Because $\mathcal{K}_t$ is injective,

$$\overline{\operatorname{Ran}(\mathcal{K}_t^*)} = (\ker \mathcal{K}_t)^\perp = L^2([0,t]).$$

Thus the first chaos generated by $\{U_r : 0 \le r \le t\}$ equals the first chaos generated by $\{B_r : 0 \le r \le t\}$. Lemma 4.4 yields (36). □

Definition 4.7 (Observed-channel filtrations)**.** For $0 \le t \le T$, define

$$\mathcal{F}_t^Y := \sigma(\mathcal{Y}_r : 0 \le r \le t), \qquad \mathcal{F}_t^{X,Y} := \sigma(X_r^{\mathrm{obs}}, \mathcal{Y}_r : 0 \le r \le t).$$

For $0 \le t < t+h \le T$, define the observed-channel prediction risks

$$\widetilde{R}_Y(t,h) := \mathbb{E}\Big[\operatorname{Var}(\Delta_h \mathcal{Y}_t \mid \mathcal{F}_t^Y)\Big], \tag{37}$$

$$\widetilde{R}_{XY}(t,h) := \mathbb{E}\Big[\operatorname{Var}(\Delta_h \mathcal{Y}_t \mid \mathcal{F}_t^{X,Y})\Big], \tag{38}$$

and the associated gain

$$\widetilde{G}_{X\to Y}(t,h) := \widetilde{R}_Y(t,h) - \widetilde{R}_{XY}(t,h). \tag{39}$$

Definition 4.8 (Past-state operators)**.** Fix $t \in (0,T]$ and write $H_t := L^2([0,t])$ and $H_t^\oplus := H_t \oplus H_t$. Define the truncated Volterra operators

$$(\mathcal{K}_{Y,t} f)(r) := \int_0^r K_Y(r,s) f(s)\ ds, \qquad (\mathcal{K}_{XY,t} g)(r) := \int_0^r K_{XY}(r,s) g(s)\ ds$$

for $0 \le r \le t$, and the observation operator

$$\mathcal{O}_t : H_t^\oplus \to H_t, \qquad \mathcal{O}_t(f,g) := \nu_Y \mathcal{K}_{Y,t} f + \eta \mathcal{K}_{XY,t} g. \tag{40}$$

Let

$$I_t^\oplus(f,g) := \int_0^t f(s)\ dW_s^\perp + \int_0^t g(s)\ dW_s^X$$

be the isonormal map on $H_t^\oplus$.

For $0 \le t < t+h \le T$, set

$$g_{Y,t,h}(s) := K_Y(t+h,s) - K_Y(t,s), \tag{41}$$

$$g_{XY,t,h}(s) := K_{XY}(t+h,s) - K_{XY}(t,s), \tag{42}$$

and define the increment test vector

$$\gamma_{t,h} := (\nu_Y g_{Y,t,h},\ \eta g_{XY,t,h}) \in H_t^{\oplus}. \tag{43}$$

Let Π_t denote the orthogonal projection in $H_t^{\oplus}$ onto

$$\overline{\mathrm{Ran}(\mathcal{O}_t^*)},$$

and let

$$\Sigma_t := I_{H_t^{\oplus}} - \Pi_t. \tag{44}$$

Remark 4.9 (Interpretation of Σ_t). The operator Σ_t is the posterior covariance operator of the hidden two-component past state $(W^{\perp}_{[0,t]}, W^X_{[0,t]})$ after observing the target past filtration $\mathcal{F}_t^Y$. The exact observed-channel gain is the quadratic form of Σ_t against the past-resolved increment vector $\gamma_{t,h}$.

Proposition 4.10 (Exact observed-channel reduction)**.** *Assume Assumptions 3.1 and 4.1. Fix $0 < t < t+h \le T$ and define the past-resolved term*

$$P_{t,h} := I_t^{\oplus}(\gamma_{t,h}) \tag{45}$$

and the future innovation

$$M_{t,h} := \nu_Y \int_t^{t+h} K_Y(t+h,s)\, dW_s^{\perp} + \eta \int_t^{t+h} K_{XY}(t+h,s)\, dW_s^X. \tag{46}$$

Then:

(i)

$$\Delta_h \mathcal{Y}_t = P_{t,h} + M_{t,h}, \tag{47}$$

with $M_{t,h}$ independent of $\mathcal{F}_t^Y$ and of $\mathcal{F}_t^{X,Y}$;

(ii)

$$\widetilde{R}_{XY}(t,h) = \nu_Y^2 \int_t^{t+h} K_Y(t+h,s)^2\, ds + \eta^2 \int_t^{t+h} K_{XY}(t+h,s)^2\, ds; \tag{48}$$

(iii) the target-past first chaos is

$$\overline{\mathrm{span}}^{L^2(\Omega)}\{\mathcal{Y}_r : 0 \le r \le t\} = I_t^{\oplus}(\overline{\mathrm{Ran}(\mathcal{O}_t^*)}); \tag{49}$$

(iv) the observed-channel prediction gain is exactly

$$\widetilde{G}_{X\to Y}(t,h) = \mathbb{E}\Big[\mathrm{Var}(P_{t,h} \mid \mathcal{F}_t^Y)\Big] = \|\Sigma_t \gamma_{t,h}\|^2_{H_t^{\oplus}}. \tag{50}$$

Proof. By the definitions of $g_{Y,t,h}$ and $g_{XY,t,h}$,

$$P_{t,h} = \nu_Y \int_0^t \big(K_Y(t+h,s) - K_Y(t,s)\big)\, dW_s^{\perp} + \eta \int_0^t \big(K_{XY}(t+h,s) - K_{XY}(t,s)\big)\, dW_s^X,$$

so (47) follows by splitting the increment $\Delta_h \mathcal{Y}_t$ into its past-resolved and future-innovation parts. The future innovation $M_{t,h}$ depends only on the Brownian increments on $(t, t+h]$, hence is independent of the sigma-field generated by

$$\{W_s^{\perp}, W_s^X : 0 \le s \le t\},$$

and therefore of both $\mathcal{F}_t^Y$ and $\mathcal{F}_t^{X,Y}$.

To prove (48), first note from Lemma 3.11 applied to the observed source channel that $r \mapsto X_r^{\mathrm{obs}}$ is continuous from $[0,t]$ into $L^2(\Omega)$. Proposition 4.6 applied to that channel therefore yields

$$\mathcal{F}_t^{X,Y} = \sigma(W_r^X, \mathcal{Y}_r : 0 \le r \le t) \qquad \mathrm{mod}\ \mathbb{P}.$$

Hence the past contagion process

$$Z_r := \int_0^r K_{XY}(r,s)\, dW_s^X, \qquad 0 \le r \le t,$$

is $\mathcal{F}_t^{X,Y}$-measurable. Since

$$\mathcal{Y}_r = \nu_Y V_r + \eta Z_r, \qquad V_r := \int_0^r K_Y(r,s)\, dW_s^{\perp},$$

it follows that the full idiosyncratic past process $\{V_r : 0 \le r \le t\}$ is also $\mathcal{F}_t^{X,Y}$-measurable. By Lemma 3.11, the map $r \mapsto V_r$ is continuous from $[0,t]$ into $L^2(\Omega)$. Applying Proposition 4.6 to the channel $\{V_r : 0 \le r \le t\}$ and the Brownian motion $W^{\perp}$, and using the injectivity of $\mathcal{K}_{Y,t}$ from Assumption 4.1, we obtain

$$\sigma(V_r : 0 \le r \le t) = \sigma(W_r^{\perp} : 0 \le r \le t) \qquad \text{mod } \mathbb{P}.$$

Hence

$$\mathcal{F}_t^{X,Y} = \sigma(W_r^{\perp}, W_r^X : 0 \le r \le t) = \mathcal{G}_t^{X,Y} \qquad \text{mod } \mathbb{P}.$$

Therefore $P_{t,h}$ is $\mathcal{F}_t^{X,Y}$-measurable. Using (47) and independence of $M_{t,h}$,

$$\operatorname{Var}(\Delta_h \mathcal{Y}_t \mid \mathcal{F}_t^{X,Y}) = \operatorname{Var}(M_{t,h}),$$

almost surely. Itô isometry gives (48).

For (49), first note from Lemma 3.11 applied to the two Volterra components of $\mathcal{Y}$ that $r \mapsto \mathcal{Y}_r$ is continuous from $[0,t]$ into $L^2(\Omega)$. For $f \in H_t$,

$$\begin{aligned}\int_0^t f(r)\mathcal{Y}_r\, dr &= \nu_Y \int_0^t f(r) \int_0^r K_Y(r,s)\, dW_s^{\perp}\, dr + \eta \int_0^t f(r) \int_0^r K_{XY}(r,s)\, dW_s^X\, dr \\ &= I_t^{\oplus}(\mathcal{O}_t^* f),\end{aligned}$$

where

$$\mathcal{O}_t^* f = (\nu_Y \mathcal{K}_{Y,t}^* f, \eta \mathcal{K}_{XY,t}^* f).$$

Lemma 4.5 therefore implies that the closed first chaos generated by the target past equals $I_t^{\oplus}(\overline{\operatorname{Ran}(\mathcal{O}_t^*)})$.

Finally, because $M_{t,h}$ is independent of $\mathcal{F}_t^Y$,

$$\operatorname{Var}(\Delta_h \mathcal{Y}_t \mid \mathcal{F}_t^Y) = \operatorname{Var}(P_{t,h} \mid \mathcal{F}_t^Y) + \operatorname{Var}(M_{t,h}).$$

Taking expectations and subtracting (48) gives

$$\widetilde{G}_{X\to Y}(t,h) = \mathbb{E}\Big[\operatorname{Var}(P_{t,h} \mid \mathcal{F}_t^Y)\Big].$$

Now $P_{t,h} = I_t^{\oplus}(\gamma_{t,h})$ lies in the hidden first chaos $I_t^{\oplus}(H_t^{\oplus})$, while the target-past first chaos is $I_t^{\oplus}(\overline{\operatorname{Ran}(\mathcal{O}_t^*)})$. By Lemma 4.4,

$$\mathcal{F}_t^Y = \sigma\big(I_t^{\oplus}(h) : h \in \overline{\operatorname{Ran}(\mathcal{O}_t^*)}\big) \qquad \text{mod } \mathbb{P}.$$

All variables are centred and jointly Gaussian. Conditioning on this sigma-field is therefore the orthogonal projection onto its closed first chaos, and

$$\mathbb{E}[P_{t,h} \mid \mathcal{F}_t^Y] = I_t^{\oplus}(\Pi_t \gamma_{t,h}),$$

and hence

$$\mathbb{E}\Big[\operatorname{Var}(P_{t,h} \mid \mathcal{F}_t^Y)\Big] = \|(I_{H_t^{\oplus}} - \Pi_t)\gamma_{t,h}\|_{H_t^{\oplus}}^2 = \|\Sigma_t \gamma_{t,h}\|_{H_t^{\oplus}}^2.$$

This proves (50). □

5. Deterministic geometry of the observed gain

Section 4 reduces observable prediction to a deterministic distance in the hidden past-state space. Variational and nullspace formulas describe the same distance. The obstruction is the failure of the target past to reproduce the enlarged oracle gain.

Definition 5.1 (Normal operator and score vector)**.** Fix $t \in (0,T]$. Define the positive self-adjoint operator

$$\mathcal{C}_t := \mathcal{O}_t\mathcal{O}_t^* = \nu_Y^2\mathcal{K}_{Y,t}\mathcal{K}_{Y,t}^* + \eta^2\mathcal{K}_{XY,t}\mathcal{K}_{XY,t}^* \tag{51}$$

on $H_t = L^2([0,t])$. For $0 \le t < t+h \le T$, define the score vector

$$r_{t,h} := \mathcal{O}_t\gamma_{t,h} = \nu_Y^2\mathcal{K}_{Y,t}g_{Y,t,h} + \eta^2\mathcal{K}_{XY,t}g_{XY,t,h}. \tag{52}$$

Proposition 5.2 (Variational representation of the observed gain)**.** *Assume Assumptions 3.1 and 4.1. Fix* $0 < t < t+h \le T$. *Then*

$$\widetilde{G}_{X\to Y}(t,h) = \inf_{f\in H_t}\left\{\nu_Y^2\|g_{Y,t,h} - \mathcal{K}_{Y,t}^*f\|_{H_t}^2 + \eta^2\|g_{XY,t,h} - \mathcal{K}_{XY,t}^*f\|_{H_t}^2\right\}. \tag{53}$$

Equivalently,

$$\widetilde{G}_{X\to Y}(t,h) = \|\gamma_{t,h}\|_{H_t^\oplus}^2 - \sup_{f\in H_t}\left\{2\langle r_{t,h}, f\rangle_{H_t} - \langle \mathcal{C}_tf, f\rangle_{H_t}\right\}. \tag{54}$$

Proof. By Proposition 4.10,

$$\widetilde{G}_{X\to Y}(t,h) = \|\Sigma_t\gamma_{t,h}\|_{H_t^\oplus}^2.$$

Since Σ_t is the orthogonal projection onto $\overline{\mathrm{Ran}(\mathcal{O}_t^*)}^\perp$, this equals the squared distance from $\gamma_{t,h}$ to $\overline{\mathrm{Ran}(\mathcal{O}_t^*)}$. Therefore

$$\widetilde{G}_{X\to Y}(t,h) = \inf_{f\in H_t}\|\gamma_{t,h} - \mathcal{O}_t^*f\|_{H_t^\oplus}^2.$$

Using (40) and (43), we obtain

$$\gamma_{t,h} - \mathcal{O}_t^*f = (\nu_Y(g_{Y,t,h} - \mathcal{K}_{Y,t}^*f),\ \eta(g_{XY,t,h} - \mathcal{K}_{XY,t}^*f)),$$

which gives (53).

Expanding the square,

$$\begin{aligned}\|\gamma_{t,h} - \mathcal{O}_t^*f\|_{H_t^\oplus}^2 &= \|\gamma_{t,h}\|_{H_t^\oplus}^2 - 2\langle\gamma_{t,h}, \mathcal{O}_t^*f\rangle_{H_t^\oplus} + \|\mathcal{O}_t^*f\|_{H_t^\oplus}^2\\ &= \|\gamma_{t,h}\|_{H_t^\oplus}^2 - 2\langle\mathcal{O}_t\gamma_{t,h}, f\rangle_{H_t} + \langle\mathcal{O}_t\mathcal{O}_t^*f, f\rangle_{H_t}\\ &= \|\gamma_{t,h}\|_{H_t^\oplus}^2 - 2\langle r_{t,h}, f\rangle_{H_t} + \langle\mathcal{C}_tf, f\rangle_{H_t}.\end{aligned}$$

Taking the infimum over f yields (54). □

Proposition 5.3 (Nullspace representation of the observed gain)**.** *Assume Assumptions 3.1 and 4.1. Fix* $0 < t < t+h \le T$. *Then*

$$\ker\mathcal{O}_t = \left\{(u,v) \in H_t^\oplus : \nu_Y\mathcal{K}_{Y,t}u + \eta\mathcal{K}_{XY,t}v = 0\right\}. \tag{55}$$

Moreover, with the convention that the supremum is zero when $\ker\mathcal{O}_t = \{0\}$,

$$\widetilde{G}_{X\to Y}(t,h) = \sup\left\{\left|\langle\gamma_{t,h}, w\rangle_{H_t^\oplus}\right|^2 : w \in \ker\mathcal{O}_t,\ \|w\|_{H_t^\oplus} = 1\right\}. \tag{56}$$

Proof. Identity (55) is the kernel condition for $\mathcal{O}_t$. Because

$$(\overline{\mathrm{Ran}(\mathcal{O}_t^*)})^\perp = \ker\mathcal{O}_t,$$

the orthogonal projection Σ_t is the projection onto $\ker\mathcal{O}_t$. Hence

$$\widetilde{G}_{X\to Y}(t,h) = \|\Sigma_t\gamma_{t,h}\|_{H_t^\oplus}^2.$$

If $\ker\mathcal{O}_t = \{0\}$, then $\Sigma_t = 0$ and the convention gives (56). Otherwise, for any orthogonal projection P on a Hilbert space and any vector x,

$$\|Px\|^2 = \sup\left\{|\langle x, w\rangle|^2 : w \in \mathrm{Ran}(P),\ \|w\| = 1\right\}.$$

Applying this with $P = \Sigma_t$ and $\mathrm{Ran}(\Sigma_t) = \ker\mathcal{O}_t$ gives (56). □

Corollary 5.4 (Zero-gain criterion)**.** *Assume Assumptions 3.1 and 4.1. Fix* $0 < t < t+h \le T$. *Then the following are equivalent:*

(i) $\widetilde{G}_{X\to Y}(t,h) = 0$;

(ii) $\gamma_{t,h} \in \overline{\mathrm{Ran}(\mathcal{O}_t^*)}$;

(iii) *there exists a sequence* (f_n) *in* H_t *such that*

$$\mathcal{K}_{Y,t}^* f_n \to g_{Y,t,h}, \qquad \eta\, \mathcal{K}_{XY,t}^* f_n \to \eta\, g_{XY,t,h}$$

in H_t.

Proof. By Proposition 4.10,

$$\widetilde{G}_{X\to Y}(t,h) = \|\Sigma_t \gamma_{t,h}\|^2_{H_t^\oplus}.$$

Hence $\widetilde{G}_{X\to Y}(t,h) = 0$ if and only if $\Sigma_t\gamma_{t,h} = 0$, which is equivalent to $\gamma_{t,h} \in \overline{\mathrm{Ran}(\mathcal{O}_t^*)}$. This proves the equivalence of (i) and (ii).

Condition (ii) is equivalent to the existence of $f_n \in H_t$ such that

$$\mathcal{O}_t^* f_n \to \gamma_{t,h} \qquad \text{in } H_t^\oplus.$$

Using the explicit form of $\mathcal{O}_t^*$ and $\gamma_{t,h}$ gives condition (iii). □

Remark 5.5 (Interpretation of the geometric reduction). Section 4 gives the probabilistic reduction. Section 5 gives the deterministic geometry. The observable gain is the distance of the increment vector $\gamma_{t,h}$ from the subspace $\overline{\mathrm{Ran}(\mathcal{O}_t^*)}$, or equivalently the energy of its component along the hidden null directions of the target-past observation operator. Asymptotic screening therefore depends on whether the near-diagonal increment vector approaches that subspace or retains a nontrivial projection on $\ker \mathcal{O}_t$ as $h \downarrow 0$.

6. Transfer-operator reduction

The observation nullspace first lives on the two-component hidden state space $H_t^\oplus$. Injectivity of the idiosyncratic Volterra operator removes one component and leaves the contagion component alone. The hidden cancellation geometry is therefore encoded by the graph of a single closed operator.

Definition 6.1 (Hidden transfer operator)**.** Assume $\eta > 0$ and fix $t \in (0,T]$. Since $\mathcal{K}_{Y,t}$ is injective, its inverse

$$\mathcal{K}_{Y,t}^{-1} : \mathrm{Ran}(\mathcal{K}_{Y,t}) \to H_t$$

is well defined. Define the domain

$$\mathcal{D}(\mathcal{S}_t) := \{v \in H_t : \mathcal{K}_{XY,t}v \in \mathrm{Ran}(\mathcal{K}_{Y,t})\}, \tag{57}$$

and the transfer operator

$$\mathcal{S}_t : \mathcal{D}(\mathcal{S}_t) \subset H_t \to H_t, \qquad \mathcal{S}_t v := \mathcal{K}_{Y,t}^{-1}\mathcal{K}_{XY,t}v. \tag{58}$$

This is the operator-theoretic form of the observable screening problem. The Volterra-equation background comes from [13]; the compactness and graph arguments below are carried in trace-ideal terms [14].

Proposition 6.2 (Closedness and graph form of the hidden nullspace)**.** *Assume Assumptions 3.1 and 4.1, and let* $\eta > 0$. *Fix* $t \in (0,T]$. *Then* $\mathcal{S}_t$ *is a linear closed operator on* H_t, *and*

$$\ker \mathcal{O}_t = \left\{ \left(-\frac{\eta}{\nu_Y}\mathcal{S}_t v,\ v \right) : v \in \mathcal{D}(\mathcal{S}_t) \right\}. \tag{59}$$

Proof. The domain $\mathcal{D}(\mathcal{S}_t)$ is linear because $\mathrm{Ran}(\mathcal{K}_{Y,t})$ is linear and both Volterra operators are linear. On this domain, $\mathcal{S}_t$ is linear by the linearity of $\mathcal{K}_{XY,t}$ and of $\mathcal{K}_{Y,t}^{-1}$ on $\mathrm{Ran}(\mathcal{K}_{Y,t})$.

To prove closedness, let $v_n \in \mathcal{D}(\mathcal{S}_t)$ satisfy

$$v_n \to v, \qquad \mathcal{S}_t v_n \to u$$

in H_t. Since $\mathcal{K}_{XY,t}$ and $\mathcal{K}_{Y,t}$ are bounded,

$$\mathcal{K}_{XY,t}v_n \to \mathcal{K}_{XY,t}v, \qquad \mathcal{K}_{Y,t}\mathcal{S}_t v_n \to \mathcal{K}_{Y,t}u$$

in H_t. The identity $\mathcal{K}_{Y,t}\mathcal{S}_t v_n = \mathcal{K}_{XY,t}v_n$ passes to the limit, so

$$\mathcal{K}_{XY,t}v = \mathcal{K}_{Y,t}u.$$

Hence $\mathcal{K}_{XY,t}v \in \mathrm{Ran}(\mathcal{K}_{Y,t})$, so $v \in \mathcal{D}(\mathcal{S}_t)$, and injectivity of $\mathcal{K}_{Y,t}$ gives $\mathcal{S}_t v = u$. Thus $\mathcal{S}_t$ is closed.

Now let $(u,v) \in H_t^{\oplus}$. By definition of $\mathcal{O}_t$,

$$(u,v) \in \ker \mathcal{O}_t \iff \nu_Y \mathcal{K}_{Y,t}u + \eta \mathcal{K}_{XY,t}v = 0.$$

Since $\eta > 0$, this is equivalent to

$$\mathcal{K}_{XY,t}v = -\frac{\nu_Y}{\eta}\mathcal{K}_{Y,t}u.$$

Therefore $v \in \mathcal{D}(\mathcal{S}_t)$ and

$$\mathcal{S}_t v = \mathcal{K}_{Y,t}^{-1}\mathcal{K}_{XY,t}v = -\frac{\nu_Y}{\eta}u,$$

so $u = -(\eta/\nu_Y)\mathcal{S}_t v$. This gives one inclusion in (59). Conversely, if $v \in \mathcal{D}(\mathcal{S}_t)$ and $u := -(\eta/\nu_Y)\mathcal{S}_t v$, then

$$\nu_Y \mathcal{K}_{Y,t}u + \eta\mathcal{K}_{XY,t}v = -\eta\mathcal{K}_{Y,t}\mathcal{S}_t v + \eta\mathcal{K}_{XY,t}v = 0,$$

so $(u,v) \in \ker \mathcal{O}_t$. This proves (59). □

Proposition 6.3 (Transfer-operator formula for the observed gain)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, and fix $0 < t < t+h \le T$. Then*

$$\widetilde{G}_{X\to Y}(t,h) = \eta^2 \sup_{v\in\mathcal{D}(\mathcal{S}_t)\setminus\{0\}} \frac{\left|\langle g_{XY,t,h}, v\rangle_{H_t} - \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t}\right|^2}{\|v\|_{H_t}^2 + \frac{\eta^2}{\nu_Y^2}\|\mathcal{S}_t v\|_{H_t}^2}, \tag{60}$$

with the convention that the supremum is 0 *if* $\mathcal{D}(\mathcal{S}_t) = \{0\}$.

Proof. If $\mathcal{D}(\mathcal{S}_t) = \{0\}$, then Proposition 6.2 gives $\ker \mathcal{O}_t = \{0\}$. Proposition 5.3 therefore yields $\widetilde{G}_{X\to Y}(t,h) = 0$, which agrees with the convention in (60).

Assume now that $\mathcal{D}(\mathcal{S}_t) \neq \{0\}$. By Proposition 6.2, every nonzero vector in $\ker \mathcal{O}_t$ can be written uniquely as

$$w_v := \left(-\frac{\eta}{\nu_Y}\mathcal{S}_t v,\ v\right), \qquad v \in \mathcal{D}(\mathcal{S}_t) \setminus \{0\}.$$

Its norm is

$$\|w_v\|_{H_t^{\oplus}}^2 = \frac{\eta^2}{\nu_Y^2}\|\mathcal{S}_t v\|_{H_t}^2 + \|v\|_{H_t}^2. \tag{61}$$

Using (43),

$$\begin{aligned}\langle \gamma_{t,h}, w_v\rangle_{H_t^{\oplus}} &= \left\langle (\nu_Y g_{Y,t,h}, \eta g_{XY,t,h}), \left(-\frac{\eta}{\nu_Y}\mathcal{S}_t v,\ v\right)\right\rangle_{H_t^{\oplus}} \\ &= \eta\Big(\langle g_{XY,t,h}, v\rangle_{H_t} - \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t}\Big).\end{aligned} \tag{62}$$

Applying Proposition 5.3 and parametrising unit vectors in $\ker \mathcal{O}_t$ by $w_v/\|w_v\|_{H_t^{\oplus}}$ gives

$$\widetilde{G}_{X\to Y}(t,h) = \sup_{v\in\mathcal{D}(\mathcal{S}_t)\setminus\{0\}} \frac{|\langle \gamma_{t,h}, w_v\rangle_{H_t^{\oplus}}|^2}{\|w_v\|_{H_t^{\oplus}}^2}.$$

Substituting (61) and (62) yields (60). □

Corollary 6.4 (Transfer-operator zero-gain criterion)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, and fix $0 < t < t+h \le T$. Then the following are equivalent:*

(i) $\widetilde{G}_{X\to Y}(t,h) = 0$*;*

(ii) for every $v \in \mathcal{D}(\mathcal{S}_t)$*,*

$$\langle g_{XY,t,h}, v\rangle_{H_t} = \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t}; \tag{63}$$

(iii) for every $(u,v) \in \ker \mathcal{O}_t$*,*

$$\nu_Y\langle g_{Y,t,h}, u\rangle_{H_t} + \eta\langle g_{XY,t,h}, v\rangle_{H_t} = 0. \tag{64}$$

Proof. If $\mathcal{D}(\mathcal{S}_t) = \{0\}$, then Proposition 6.3 gives $\widetilde{G}_{X\to Y}(t,h) = 0$, and both (ii) and (iii) are vacuous. So suppose $\mathcal{D}(\mathcal{S}_t) \neq \{0\}$.

By Proposition 6.3, $\widetilde{G}_{X\to Y}(t,h) = 0$ if and only if

$$\langle g_{XY,t,h}, v\rangle_{H_t} - \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t} = 0$$

for every $v \in \mathcal{D}(\mathcal{S}_t)$, which proves the equivalence of (i) and (ii).

For (ii) versus (iii), Proposition 6.2 says that every $(u,v) \in \ker \mathcal{O}_t$ is of the form

$$(u,v) = \left(-\frac{\eta}{\nu_Y}\mathcal{S}_t v,\ v\right)$$

with $v \in \mathcal{D}(\mathcal{S}_t)$. Substituting this identity into (64) gives exactly (63), so (ii) and (iii) are equivalent. □

Remark 6.5 (Interpretation of the transfer-operator reduction)*.* The operator $\mathcal{S}_t$ maps a contagion-side hidden direction into the unique idiosyncratic hidden direction that cancels it in the target past. Thus the observable gain is controlled by a single mismatch functional,

$$v \mapsto \langle g_{XY,t,h}, v\rangle_{H_t} - \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t},$$

on the graph domain of $\mathcal{S}_t$. The asymptotic analysis therefore reduces to the near-diagonal action of $\mathcal{S}_t$ on the singular increment profiles $g_{Y,t,h}$ and $g_{XY,t,h}$.

7. Graph-space blow-up for the unbounded regime

The bounded-transfer case covers the regime in which the cancellation operator acts on all of H_t. Rougher contagion changes the space. The transfer operator is expected to be unbounded, the ambient object becomes the graph space of $\mathcal{S}_t$, and the observable gain becomes the dual norm of a graph-space functional. The near-diagonal increment profiles are the local objects that survive the blow-up.

Definition 7.1 (Graph space and graph functional)**.** Assume $\eta > 0$ and fix $t \in (0,T]$. Let

$$\mathfrak{G}_t := \mathcal{D}(\mathcal{S}_t)$$

equipped with the graph inner product

$$\langle v, w\rangle_{\mathfrak{G}_t} := \langle v, w\rangle_{H_t} + \frac{\eta^2}{\nu_Y^2}\langle \mathcal{S}_t v, \mathcal{S}_t w\rangle_{H_t}, \qquad v, w \in \mathfrak{G}_t. \tag{65}$$

For $0 \le t < t+h \le T$, define the graph functional

$$\mathcal{F}_{t,h}(v) := \langle g_{XY,t,h}, v\rangle_{H_t} - \langle g_{Y,t,h}, \mathcal{S}_t v\rangle_{H_t}, \qquad v \in \mathfrak{G}_t. \tag{66}$$

Proposition 7.2 (Exact graph-space dual formula)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, and fix $0 < t < t+h \le T$. Then:*

(i) $(\mathfrak{G}_t, \langle\cdot,\cdot\rangle_{\mathfrak{G}_t})$ is a Hilbert space;
(ii) $\mathcal{F}_{t,h}$ is a continuous linear functional on $\mathfrak{G}_t$;
(iii) the observed gain satisfies

$$\widetilde{G}_{X\to Y}(t,h) = \eta^2 \|\mathcal{F}_{t,h}\|^2_{\mathfrak{G}_t^*}. \tag{67}$$

Proof. Because $\mathcal{S}_t$ is closed by Proposition 6.2, its graph is closed in $H_t^{\oplus}$. Therefore the graph norm

$$\|v\|^2_{\mathfrak{G}_t} = \|v\|^2_{H_t} + \frac{\eta^2}{\nu_Y^2}\|\mathcal{S}_t v\|^2_{H_t}$$

makes $\mathfrak{G}_t$ into a Hilbert space, proving (i).

For $v \in \mathfrak{G}_t$, Cauchy–Schwarz gives

$$\begin{aligned}|\mathcal{F}_{t,h}(v)| &\le \|g_{XY,t,h}\|_{H_t}\|v\|_{H_t} + \|g_{Y,t,h}\|_{H_t}\|\mathcal{S}_t v\|_{H_t} \\ &\le \left(\|g_{XY,t,h}\|_{H_t} + \frac{\nu_Y}{\eta}\|g_{Y,t,h}\|_{H_t}\right)\|v\|_{\mathfrak{G}_t},\end{aligned}$$

so $\mathcal{F}_{t,h} \in \mathfrak{G}_t^*$, proving (ii).

Finally, if $\mathfrak{G}_t = \{0\}$, then Proposition 6.3 gives $\widetilde{G}_{X\to Y}(t,h) = 0$ and $\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = 0$. Otherwise, Proposition 6.3 states that

$$\widetilde{G}_{X\to Y}(t,h) = \eta^2 \sup_{v\in\mathfrak{G}_t\setminus\{0\}} \frac{|\mathcal{F}_{t,h}(v)|^2}{\|v\|_{\mathfrak{G}_t}^2},$$

which is (67). This proves (iii). □

Definition 7.3 (Dilation and universal profile)**.** For $t \in (0,T]$ and $h \in (0,t]$, define the dilation map

$$(\mathcal{D}_{t,h}f)(u) := h^{1/2} f(t-hu)\mathbf{1}_{(0,t/h)}(u), \qquad u > 0, \tag{68}$$

from $L^2([0,t])$ into $L^2(\mathbb{R}_+)$.

For $H \in (0, \frac{1}{2})$, define the universal profile

$$\psi_H(u) := (1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}, \qquad u > 0, \tag{69}$$

viewed as an element of $L^2(\mathbb{R}_+)$. This is legitimate by Lemma 3.5.

Lemma 7.4 (Dilation isometry)**.** *Fix $t \in (0,T]$ and $h \in (0,t]$. Then $\mathcal{D}_{t,h} : H_t \to L^2(\mathbb{R}_+)$ is a linear isometric embedding. For every $f, g \in H_t$,*

$$\langle \mathcal{D}_{t,h}f, \mathcal{D}_{t,h}g\rangle_{L^2(\mathbb{R}_+)} = \langle f, g\rangle_{H_t}. \tag{70}$$

Proof. Linearity follows from the definition of $\mathcal{D}_{t,h}$. For $f, g \in H_t$, the change of variables $s = t - hu$ yields

$$\begin{aligned}\langle \mathcal{D}_{t,h}f, \mathcal{D}_{t,h}g\rangle_{L^2(\mathbb{R}_+)} &= \int_0^{t/h} h\, f(t-hu)g(t-hu)\, du \\ &= \int_0^t f(s)g(s)\, ds \\ &= \langle f, g\rangle_{H_t}.\end{aligned}$$

This proves (70). □

Proposition 7.5 (Universal profile limit for the increment kernels)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$ and let $i \in \{Y, XY\}$. Define*

$$\widehat{g}_{i,t,h} := h^{-H_i}\mathcal{D}_{t,h}g_{i,t,h} \in L^2(\mathbb{R}_+), \qquad 0 < h \le t. \tag{71}$$

Then

$$\widehat{g}_{i,t,h} \longrightarrow L_i(t,t)\psi_{H_i} \qquad \text{in } L^2(\mathbb{R}_+) \tag{72}$$

as $h \downarrow 0$.

Proof. Fix $i \in \{Y, XY\}$, write $H := H_i$, $L := L_i$, and set $\alpha := H - \frac{1}{2} \in (-\frac{1}{2}, 0)$. For $u \in (0, t/h)$ one has

$$\begin{aligned}\widehat{g}_{i,t,h}(u) &= h^{\frac{1}{2}-H}\Big(K_i(t+h, t-hu) - K_i(t, t-hu)\Big) \\ &= (1+u)^\alpha L(t+h, t-hu) - u^\alpha L(t, t-hu).\end{aligned} \tag{73}$$

Also, $\widehat{g}_{i,t,h}(u) = 0$ for $u \ge t/h$.

Let

$$u_h := h^{-1/2}.$$

We split

$$\|\widehat{g}_{i,t,h} - L(t,t)\psi_H\|_{L^2(\mathbb{R}_+)}^2 \le I_h^{\text{near}} + 2I_h^{\text{far}} + 2I_h^{\text{tail}},$$

where

$$I_h^{\mathrm{near}} := \int_0^{u_h} \left|\widehat{g}_{i,t,h}(u) - L(t,t)\psi_H(u)\right|^2 du,$$
$$I_h^{\mathrm{far}} := \int_{u_h}^{t/h} \left|\widehat{g}_{i,t,h}(u)\right|^2 du,$$
$$I_h^{\mathrm{tail}} := L(t,t)^2 \int_{u_h}^{\infty} |\psi_H(u)|^2 du.$$

Near-diagonal region. For $u \in [0, u_h]$, one has $0 \le hu \le h^{1/2}$. Since L is C^1 near the diagonal, there exists $C < \infty$ such that

$$|L(t+h, t-hu) - L(t,t)| + |L(t, t-hu) - L(t,t)| \le C(h + hu) \le 2Ch^{1/2}.$$

Hence by (73),

$$\widehat{g}_{i,t,h}(u) - L(t,t)\psi_H(u) = (1+u)^\alpha \Delta_{1,h}(u) - u^\alpha \Delta_{2,h}(u),$$

with $|\Delta_{1,h}(u)| + |\Delta_{2,h}(u)| \le 2Ch^{1/2}$. Therefore

$$\left|\widehat{g}_{i,t,h}(u) - L(t,t)\psi_H(u)\right| \le Ch^{1/2}\big((1+u)^\alpha + u^\alpha\big),$$

and so

$$I_h^{\mathrm{near}} \lesssim h \int_0^{u_h} \big((1+u)^{2H-1} + u^{2H-1}\big)\, du = O(h^{1-H}) \longrightarrow 0.$$

Far region. Since $h^{1/2} \ge h$ for $0 < h \le 1$, Lemma 3.10 yields a constant $C < \infty$ such that for all sufficiently small h and all $s \in [0, t - h^{1/2}]$,

$$|K_i(t+h, s) - K_i(t,s)| \le Ch\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big).$$

Since $u \mapsto t - hu$ maps $[u_h, t/h]$ onto $[0, t - h^{1/2}]$, it follows that

$$\begin{aligned}
I_h^{\mathrm{far}} &= \int_{u_h}^{t/h} |\widehat{g}_{i,t,h}(u)|^2 du \\
&= h^{-2H} \int_0^{t-h^{1/2}} \big(K_i(t+h,s) - K_i(t,s)\big)^2 ds \\
&\lesssim h^{2-2H} \int_0^{t-h^{1/2}} \big((t-s)^{2H-3} + (t-s)^{2H-1}\big)\, ds \\
&= O(h^{1-H}) + O(h^{2-2H}) \longrightarrow 0.
\end{aligned}$$

Tail of the limit profile. Since $\psi_H \in L^2(\mathbb{R}_+)$ by Lemma 3.5, one has

$$I_h^{\mathrm{tail}} \longrightarrow 0 \qquad \text{as } h \downarrow 0.$$

Combining the three terms proves (72). □

Corollary 7.6 (Profile pairing limit)**.** *Under the assumptions of Proposition 7.5, for every $\phi \in L^2(\mathbb{R}_+)$ one has*

$$\langle \widehat{g}_{i,t,h}, \phi \rangle_{L^2(\mathbb{R}_+)} \longrightarrow L_i(t,t)\, \langle \psi_{H_i}, \phi \rangle_{L^2(\mathbb{R}_+)} \tag{74}$$

as $h \downarrow 0$.

Proof. By Proposition 7.5,

$$\widehat{g}_{i,t,h} - L_i(t,t)\psi_{H_i} \to 0 \qquad \text{in } L^2(\mathbb{R}_+).$$

For fixed $\phi \in L^2(\mathbb{R}_+)$, Cauchy–Schwarz gives

$$\left|\langle \widehat{g}_{i,t,h} - L_i(t,t)\psi_{H_i}, \phi \rangle_{L^2(\mathbb{R}_+)}\right| \le \|\widehat{g}_{i,t,h} - L_i(t,t)\psi_{H_i}\|_{L^2(\mathbb{R}_+)} \, \|\phi\|_{L^2(\mathbb{R}_+)},$$

and the right-hand side tends to zero. □

8. Fractional model cancellation

The profile block identifies the singular shapes carried by the increment kernels. Cancellation requires an operator with the same tangent scale. On the half-line the natural candidate is a fractional derivative of order $H_Y - H_{XY}$. The exact model identity below is the cancellation law that any graph blow-up must satisfy; the transfer-operator proof of that blow-up remains separate.

Definition 8.1 (Liouville–Weyl fractional derivatives)**.** Let $\beta \in (0,1)$. For $\phi \in C_c^\infty(\mathbb{R})$, define

$$(\mathfrak{D}_+^\beta \phi)(x) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^{x} (x-y)^{-\beta} \phi(y)\, dy, \tag{75}$$

$$(\mathfrak{D}_-^\beta \phi)(x) := -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{x}^{\infty} (y-x)^{-\beta} \phi(y)\, dy. \tag{76}$$

We extend $\mathfrak{D}_+^\beta$ to distributions by duality:

$$\langle \mathfrak{D}_+^\beta U, \phi\rangle := \langle U, \mathfrak{D}_-^\beta \phi\rangle, \qquad U \in \mathcal{D}'(\mathbb{R}), \quad \phi \in C_c^\infty(\mathbb{R}). \tag{77}$$

Finally, for $a \in \mathbb{R}$, let τ_a denote translation by

$$(\tau_a f)(x) := f(x+a).$$

Lemma 8.2 (Adjointness and translation invariance)**.** *Let $\beta \in (0,1)$.*

(i) For every $f, g \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} (\mathfrak{D}_+^\beta f)(x) g(x)\, dx = \int_{\mathbb{R}} f(x) (\mathfrak{D}_-^\beta g)(x)\, dx. \tag{78}$$

(ii) For every $a \in \mathbb{R}$ and every $f \in C_c^\infty(\mathbb{R})$,

$$\mathfrak{D}_\pm^\beta (\tau_a f) = \tau_a (\mathfrak{D}_\pm^\beta f). \tag{79}$$

Proof. For $f \in C_c^\infty(\mathbb{R})$, set

$$(\mathfrak{I}_+^{1-\beta} f)(x) := \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{x} (x-y)^{-\beta} f(y)\, dy,$$

so that $\mathfrak{D}_+^\beta f = (\mathfrak{I}_+^{1-\beta} f)'$. Likewise, for $g \in C_c^\infty(\mathbb{R})$,

$$(\mathfrak{I}_-^{1-\beta} g)(x) := \frac{1}{\Gamma(1-\beta)} \int_{x}^{\infty} (y-x)^{-\beta} g(y)\, dy,$$

so that $\mathfrak{D}_-^\beta g = -(\mathfrak{I}_-^{1-\beta} g)'$.

Because f and g are compactly supported, $\mathfrak{I}_+^{1-\beta} f$ and $\mathfrak{I}_-^{1-\beta} g$ are absolutely continuous and vanish at $\pm\infty$. Hence integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}} (\mathfrak{D}_+^\beta f)(x) g(x)\, dx &= -\int_{\mathbb{R}} (\mathfrak{I}_+^{1-\beta} f)(x) g'(x)\, dx \\ &= -\frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}} \int_{-\infty}^{x} (x-y)^{-\beta} f(y) g'(x)\, dy\, dx.\end{aligned}$$

Fubini's theorem applies because the integrand is absolutely integrable. Therefore

$$\int_{\mathbb{R}} (\mathfrak{D}_+^\beta f)(x) g(x)\, dx = -\frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}} f(y) \left(\int_{y}^{\infty} (x-y)^{-\beta} g'(x)\, dx \right) dy.$$

After the change of variables $x = y + r$,

$$\frac{1}{\Gamma(1-\beta)} \int_{y}^{\infty} (x-y)^{-\beta} g'(x)\, dx = \frac{d}{dy} (\mathfrak{I}_-^{1-\beta} g)(y),$$

so the last display equals

$$\int_{\mathbb{R}} f(y) (-(\mathfrak{I}_-^{1-\beta} g)'(y))\, dy = \int_{\mathbb{R}} f(y) (\mathfrak{D}_-^\beta g)(y)\, dy,$$

which proves (78).

For translation invariance, let $a \in \mathbb{R}$. A change of variables gives

$$
\begin{aligned}
(\mathfrak{D}_+^\beta(\tau_a f))(x) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^{x+a} (x+a-z)^{-\beta} f(z)\, dz \\
&= (\tau_a \mathfrak{D}_+^\beta f)(x).
\end{aligned}
$$

The proof for $\mathfrak{D}_-^\beta$ is identical.

By duality, (79) extends from test functions to distributions. □

Proposition 8.3 (Fractional derivative of the one-sided power kernel)**.** *Let $H \in (0, \frac{1}{2})$ and $\beta \in (0, H + \frac{1}{2})$. Define*

$$
k_H(x) := x_+^{H-\frac{1}{2}}, \qquad x \in \mathbb{R}. \tag{80}
$$

Then, in $\mathcal{D}'(\mathbb{R})$,

$$
\mathfrak{D}_+^\beta k_H = \frac{\Gamma(H+\frac{1}{2})}{\Gamma(H-\beta+\frac{1}{2})}\, k_{H-\beta}. \tag{81}
$$

Proof. Write $\lambda := H + \frac{1}{2} \in (0,1)$. For $x < 0$, both sides of (81) are zero. For $x > 0$, one has

$$
\begin{aligned}
(\mathfrak{I}_+^{1-\beta} k_H)(x) &= \frac{1}{\Gamma(1-\beta)} \int_0^x (x-y)^{-\beta} y^{\lambda-1}\, dy \\
&= \frac{x^{\lambda-\beta}}{\Gamma(1-\beta)} \int_0^1 (1-r)^{-\beta} r^{\lambda-1}\, dr \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+1-\beta)} x^{\lambda-\beta},
\end{aligned}
$$

by the Beta-function identity. Differentiating yields

$$
(\mathfrak{D}_+^\beta k_H)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\beta)} x^{\lambda-\beta-1} = \frac{\Gamma(H+\frac{1}{2})}{\Gamma(H-\beta+\frac{1}{2})} x^{H-\beta-\frac{1}{2}}, \qquad x > 0.
$$

Both sides are locally integrable on $\mathbb{R}$, so the almost-everywhere identity extends to the distributional identity (81). □

Proposition 8.4 (Exact fractional cancellation of the universal profiles)**.** *Assume*

$$
0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY} \in (0, H_Y).
$$

Fix $t \in (0,T)$ and define the model transfer coefficient

$$
\mathbf{c}_t := \frac{L_{XY}(t,t)}{L_Y(t,t)} \frac{\Gamma(H_{XY}+\frac{1}{2})}{\Gamma(H_Y+\frac{1}{2})}. \tag{82}
$$

For $\phi \in C_c^\infty((0,\infty))$, set

$$
(\mathcal{T}_{t,\beta}^{\mathrm{mod}} \phi)(u) := \mathbf{c}_t\, (\mathfrak{D}_-^\beta \phi)(u), \qquad u > 0. \tag{83}
$$

Then for every $\phi \in C_c^\infty((0,\infty))$,

$$
L_{XY}(t,t) \langle \psi_{H_{XY}}, \phi \rangle_{L^2(\mathbb{R}_+)} = L_Y(t,t) \left\langle \psi_{H_Y}, \mathcal{T}_{t,\beta}^{\mathrm{mod}} \phi \right\rangle_{L^2(\mathbb{R}_+)}. \tag{84}
$$

Proof. For $H \in (0, \frac{1}{2})$ and $u > 0$, (69) gives

$$
\psi_H(u) = k_H(u+1) - k_H(u) = ((\tau_1 - I) k_H)(u),
$$

with k_H as in (80). By Proposition 8.3 and the translation invariance from Lemma 8.2,

$$
\begin{aligned}
\mathfrak{D}_+^\beta \psi_{H_Y} &= (\tau_1 - I) \mathfrak{D}_+^\beta k_{H_Y} \\
&= \frac{\Gamma(H_Y+\frac{1}{2})}{\Gamma(H_Y-\beta+\frac{1}{2})} (\tau_1 - I) k_{H_Y-\beta} \\
&= \frac{\Gamma(H_Y+\frac{1}{2})}{\Gamma(H_{XY}+\frac{1}{2})} \psi_{H_{XY}}
\end{aligned}
$$

in $\mathcal{D}'((0,\infty))$.

Now let $\phi \in C_c^\infty((0,\infty))$. Since ϕ is compactly supported away from the boundary, $\mathfrak{D}_-^\beta \phi \in L^2(\mathbb{R}_+)$. Hence the first and last pairings below are ordinary L^2 integrals, while the middle one is distributional. Using the duality (77),

$$\begin{aligned}\left\langle \psi_{H_Y}, \mathcal{T}_{t,\beta}^{\mathrm{mod}}\phi \right\rangle_{L^2(\mathbb{R}_+)} &= \mathbf{c}_t \langle \psi_{H_Y}, \mathfrak{D}_-^\beta \phi\rangle \\ &= \mathbf{c}_t \langle \mathfrak{D}_+^\beta \psi_{H_Y}, \phi\rangle \\ &= \mathbf{c}_t \frac{\Gamma(H_Y+\frac{1}{2})}{\Gamma(H_{XY}+\frac{1}{2})} \langle \psi_{H_{XY}}, \phi\rangle_{L^2(\mathbb{R}_+)} \\ &= \frac{L_{XY}(t,t)}{L_Y(t,t)} \langle \psi_{H_{XY}}, \phi\rangle_{L^2(\mathbb{R}_+)},\end{aligned}$$

which is exactly (84). □

Corollary 8.5 (Vanishing of the model leading functional)**.** *Under the assumptions of Proposition 8.4, define*

$$\mathcal{F}_{t,\beta}^{\mathrm{mod}}(\phi) := L_{XY}(t,t)\langle \psi_{H_{XY}}, \phi\rangle_{L^2(\mathbb{R}_+)} - L_Y(t,t)\left\langle \psi_{H_Y}, \mathcal{T}_{t,\beta}^{\mathrm{mod}}\phi\right\rangle_{L^2(\mathbb{R}_+)}, \tag{85}$$

for $\phi \in C_c^\infty((0,\infty))$. Then

$$\mathcal{F}_{t,\beta}^{\mathrm{mod}}(\phi) = 0 \qquad \textit{for every } \phi \in C_c^\infty((0,\infty)).$$

Proof. The functional $\mathcal{F}_{t,\beta}^{\mathrm{mod}}$ is the difference between the two sides of (84). Proposition 8.4 identifies those sides for every $\phi \in C_c^\infty((0,\infty))$, so the difference is zero. □

9. Directional cancellation under fractional blow-up

The model identity determines the only plausible leading-order cancellation law. Localized smooth profiles transfer that identity back to the original graph functional. The conclusion is directional; uniform control on the graph unit ball remains a separate problem.

Definition 9.1 (Localized profile lift)**.** Fix $t \in (0,T)$ and $h \in (0,t]$. For $\phi \in L^2(\mathbb{R}_+)$, define

$$(\mathcal{L}_{t,h}\phi)(s) := h^{-1/2}\phi\left(\frac{t-s}{h}\right)\mathbf{1}_{[0,t]}(s), \qquad 0 \le s \le t. \tag{86}$$

Lemma 9.2 (Dilation-lift duality)**.** *Fix $t \in (0,T)$ and let $\phi \in C_c^\infty((0,\infty))$. Then for all sufficiently small $h>0$ such that* $\operatorname{supp}(\phi) \subset (0,t/h)$*, one has:*

(i) $\mathcal{L}_{t,h}\phi \in H_t$ and

$$\|\mathcal{L}_{t,h}\phi\|_{H_t} = \|\phi\|_{L^2(\mathbb{R}_+)};$$

(ii) for every $f \in H_t$,

$$\langle f, \mathcal{L}_{t,h}\phi\rangle_{H_t} = \langle \mathcal{D}_{t,h}f, \phi\rangle_{L^2(\mathbb{R}_+)}; \tag{87}$$

(iii) for each $i \in \{Y, XY\}$,

$$h^{-H_i}\langle g_{i,t,h}, \mathcal{L}_{t,h}\phi\rangle_{H_t} = \langle \widehat{g}_{i,t,h}, \phi\rangle_{L^2(\mathbb{R}_+)}. \tag{88}$$

Proof. Choose $h>0$ so small that $\operatorname{supp}(\phi) \subset (0,t/h)$. Then

$$\|\mathcal{L}_{t,h}\phi\|_{H_t}^2 = \int_0^t h^{-1}\left|\phi\left(\frac{t-s}{h}\right)\right|^2 ds = \int_0^{t/h} |\phi(u)|^2\,du = \|\phi\|_{L^2(\mathbb{R}_+)}^2,$$

because ϕ vanishes outside $(0,t/h)$. This proves (i).

For $f \in H_t$, the change of variables $u=(t-s)/h$ gives

$$\begin{aligned}\langle f, \mathcal{L}_{t,h}\phi\rangle_{H_t} &= \int_0^t f(s)h^{-1/2}\phi\left(\frac{t-s}{h}\right)\,ds \\ &= \int_0^{t/h} h^{1/2} f(t-hu)\phi(u)\,du \\ &= \langle \mathcal{D}_{t,h}f, \phi\rangle_{L^2(\mathbb{R}_+)},\end{aligned}$$

which proves (ii). Finally, substituting $f = g_{i,t,h}$ into (87) and using (71) yields (iii). □

Definition 9.3 (Localized fractional blow-up at time t)**.** Assume

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

We say that the hidden transfer operator has *localized fractional blow-up at time t* if, for every $\phi \in C_c^\infty((0,\infty))$, there exists $h_\phi \in (0,t]$ such that

$$\mathcal{L}_{t,h}\phi \in \mathcal{D}(\mathcal{S}_t) \qquad \text{for all } 0 < h \le h_\phi,$$

and

$$h^\beta \mathcal{D}_{t,h}(\mathcal{S}_t\mathcal{L}_{t,h}\phi) \longrightarrow \mathcal{T}^{\mathrm{mod}}_{t,\beta}\phi \qquad \text{in } L^2(\mathbb{R}_+) \tag{89}$$

as $h \downarrow 0$.

Theorem 9.4 (Directional leading-order cancellation)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, and fix $t \in (0,T)$. Suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}$$

and that $\mathcal{S}_t$ has localized fractional blow-up at time t in the sense of Definition 9.3. Then for every $\phi \in C_c^\infty((0,\infty))$,

$$h^{-H_{XY}}\mathcal{F}_{t,h}(\mathcal{L}_{t,h}\phi) \longrightarrow 0 \qquad \text{as } h \downarrow 0. \tag{90}$$

Proof. Fix $\phi \in C_c^\infty((0,\infty))$ and let $h > 0$ be small enough that $\mathcal{L}_{t,h}\phi \in \mathcal{D}(\mathcal{S}_t)$. Set

$$v_h := \mathcal{L}_{t,h}\phi, \qquad z_h := h^\beta \mathcal{D}_{t,h}(\mathcal{S}_t v_h) \in L^2(\mathbb{R}_+).$$

By (66), Lemma 9.2, and the identity $H_{XY} = H_Y - \beta$,

$$\begin{aligned} h^{-H_{XY}}\mathcal{F}_{t,h}(v_h) &= h^{-H_{XY}}\langle g_{XY,t,h}, v_h\rangle_{H_t} - h^{-H_{XY}}\langle g_{Y,t,h}, \mathcal{S}_t v_h\rangle_{H_t} \\ &= \langle \widehat{g}_{XY,t,h}, \phi\rangle_{L^2(\mathbb{R}_+)} - h^{-H_Y+\beta}\langle g_{Y,t,h}, \mathcal{S}_t v_h\rangle_{H_t}. \end{aligned}$$

Applying Lemma 7.4 to the second term and using the definitions of $\widehat{g}_{Y,t,h}$ and z_h gives

$$h^{-H_Y+\beta}\langle g_{Y,t,h}, \mathcal{S}_t v_h\rangle_{H_t} = \Big\langle h^{-H_Y}\mathcal{D}_{t,h} g_{Y,t,h}, h^\beta \mathcal{D}_{t,h}(\mathcal{S}_t v_h)\Big\rangle_{L^2(\mathbb{R}_+)} = \langle \widehat{g}_{Y,t,h}, z_h\rangle_{L^2(\mathbb{R}_+)},$$

so

$$h^{-H_{XY}}\mathcal{F}_{t,h}(v_h) = \langle \widehat{g}_{XY,t,h}, \phi\rangle_{L^2(\mathbb{R}_+)} - \langle \widehat{g}_{Y,t,h}, z_h\rangle_{L^2(\mathbb{R}_+)}.$$

By Corollary 7.6,

$$\langle \widehat{g}_{XY,t,h}, \phi\rangle_{L^2(\mathbb{R}_+)} \longrightarrow L_{XY}(t,t)\langle \psi_{H_{XY}}, \phi\rangle_{L^2(\mathbb{R}_+)}.$$

Also, Proposition 7.5 gives

$$\widehat{g}_{Y,t,h} \to L_Y(t,t)\psi_{H_Y} \qquad \text{in } L^2(\mathbb{R}_+),$$

while Definition 9.3 gives

$$z_h \to \mathcal{T}^{\mathrm{mod}}_{t,\beta}\phi \qquad \text{in } L^2(\mathbb{R}_+).$$

Since (z_h) is therefore bounded in $L^2(\mathbb{R}_+)$,

$$\begin{aligned} &\big|\langle \widehat{g}_{Y,t,h}, z_h\rangle_{L^2(\mathbb{R}_+)} - L_Y(t,t)\langle \psi_{H_Y}, \mathcal{T}^{\mathrm{mod}}_{t,\beta}\phi\rangle_{L^2(\mathbb{R}_+)}\big| \\ &\quad \le \|\widehat{g}_{Y,t,h} - L_Y(t,t)\psi_{H_Y}\|_{L^2(\mathbb{R}_+)}\|z_h\|_{L^2(\mathbb{R}_+)} \\ &\qquad + L_Y(t,t)\|\psi_{H_Y}\|_{L^2(\mathbb{R}_+)}\|z_h - \mathcal{T}^{\mathrm{mod}}_{t,\beta}\phi\|_{L^2(\mathbb{R}_+)}, \end{aligned}$$

and the right-hand side tends to 0. Hence

$$\langle \widehat{g}_{Y,t,h}, z_h\rangle_{L^2(\mathbb{R}_+)} \longrightarrow L_Y(t,t)\langle \psi_{H_Y}, \mathcal{T}^{\mathrm{mod}}_{t,\beta}\phi\rangle_{L^2(\mathbb{R}_+)}.$$

Applying Proposition 8.4 to the limiting terms gives (90). □

10. Uniform suppression from near-diagonal coercivity

Directional cancellation does not control the graph-space supremum in Proposition 7.2. A separate coercive input gives the first uniform theorem. If graph-unit vectors cannot concentrate too much L^2 mass inside the $h^{1/2}$ boundary layer near t, the rough contagion increment loses its oracle scale uniformly. The far part of that increment is already of higher order and needs no further structure.

Definition 10.1 (Near/far localization on H_t)**.** Fix $t \in (0,T)$ and $h \in (0,t]$. Define the orthogonal projections

$$(\Pi^{\mathrm{near}}_{t,h} f)(s) := f(s)\mathbf{1}_{[t-h^{1/2},\,t]}(s), \tag{91}$$

$$(\Pi^{\mathrm{far}}_{t,h} f)(s) := f(s)\mathbf{1}_{[0,\,t-h^{1/2})}(s), \tag{92}$$

for $f \in H_t = L^2([0,t])$.

Lemma 10.2 (Far-zone estimate for the increment kernels)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$ and let $i \in \{Y, XY\}$. Then*

$$\|\Pi^{\mathrm{far}}_{t,h} g_{i,t,h}\|_{H_t} = O(h^{\frac{1+H_i}{2}}) \qquad \text{as } h \downarrow 0. \tag{93}$$

Proof. Write $H := H_i$. By Lemma 3.10, there exist constants $C < \infty$ and $h_0 \in (0,t]$ such that for all $0 < h \le h_0$ and all $s \in [0, t-h^{1/2}]$,

$$|g_{i,t,h}(s)| \le Ch\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big).$$

Therefore

$$\begin{aligned}\|\Pi^{\mathrm{far}}_{t,h} g_{i,t,h}\|^2_{H_t} &= \int_0^{t-h^{1/2}} |g_{i,t,h}(s)|^2\, ds \\ &\lesssim h^2 \int_0^{t-h^{1/2}} \big((t-s)^{2H-3} + (t-s)^{2H-1}\big)\, ds.\end{aligned}$$

With the change of variables $u = t-s$, this becomes

$$\|\Pi^{\mathrm{far}}_{t,h} g_{i,t,h}\|^2_{H_t} \lesssim h^2 \int_{h^{1/2}}^t (u^{2H-3} + u^{2H-1})\, du.$$

Since $H \in (0, \frac{1}{2})$,

$$\int_{h^{1/2}}^t u^{2H-3}\, du = O(h^{H-1}), \qquad \int_{h^{1/2}}^t u^{2H-1}\, du = O(1),$$

and hence

$$\|\Pi^{\mathrm{far}}_{t,h} g_{i,t,h}\|^2_{H_t} = O(h^{1+H}) + O(h^2) = O(h^{1+H}).$$

Taking square roots proves (93). □

11. Exact Volterra localization of the transfer operator

The boundary-layer mechanism is still expressed through cutoff and blow-up hypotheses. The Volterra structure replaces part of that abstraction by exact local operators. Causality makes right-end boundary cutoffs stable in graph norm. After local dilation to the unit interval, the true transfer operator induces a rescaled family whose kernels converge to the frozen fractional model. The problem becomes local transfer convergence.

Definition 11.1 (Boundary-layer subspace and unit-interval dilation)**.** Fix $t \in (0,T)$, $\alpha \in (0,1)$, and $h \in (0,t]$. Define the boundary-layer subspace

$$H^{\mathrm{near}}_{t,h,\alpha} := \{f \in H_t : \mathrm{supp}(f) \subset [t-h^\alpha, t]\}.$$

Let

$$E := L^2([0,1]).$$

Define the unit-interval dilation and lift by

$$(\mathcal{D}_{t,h}^{(\alpha)}f)(u) := h^{\alpha/2}f(t-h^\alpha u), \qquad 0 \le u \le 1, \tag{94}$$

$$(\mathcal{L}_{t,h}^{(\alpha)}\phi)(s) := h^{-\alpha/2}\phi\Big(\frac{t-s}{h^\alpha}\Big)\,\mathbf{1}_{[t-h^\alpha,t]}(s), \qquad 0 \le s \le t, \tag{95}$$

for $f \in H_{t,h,\alpha}^{\mathrm{near}}$ and $\phi \in E$.

Lemma 11.2 (Unit-interval dilation isometry)**.** *Fix $t \in (0,T)$, $\alpha \in (0,1)$, and $h \in (0,t]$. Then*

$$\mathcal{D}_{t,h}^{(\alpha)} : H_{t,h,\alpha}^{\mathrm{near}} \to E$$

is an isometric isomorphism with inverse $\mathcal{L}_{t,h}^{(\alpha)}$. In particular, for every $f \in H_{t,h,\alpha}^{\mathrm{near}}$ and $\phi \in E$,

$$\|\mathcal{D}_{t,h}^{(\alpha)}f\|_E = \|f\|_{H_t}, \tag{96}$$

$$\langle f, \mathcal{L}_{t,h}^{(\alpha)}\phi\rangle_{H_t} = \langle \mathcal{D}_{t,h}^{(\alpha)}f, \phi\rangle_E. \tag{97}$$

Proof. The change of variables $s = t - h^\alpha u$ gives

$$\|\mathcal{D}_{t,h}^{(\alpha)}f\|_E^2 = \int_0^1 h^\alpha |f(t-h^\alpha u)|^2\,du = \int_{t-h^\alpha}^t |f(s)|^2\,ds = \|f\|_{H_t}^2,$$

because f is supported in $[t-h^\alpha, t]$. The same change of variables gives (97). The definitions give $\mathcal{D}_{t,h}^{(\alpha)}\mathcal{L}_{t,h}^{(\alpha)} = I_E$ and $\mathcal{L}_{t,h}^{(\alpha)}\mathcal{D}_{t,h}^{(\alpha)} = I$ on $H_{t,h,\alpha}^{\mathrm{near}}$. □

Proposition 11.3 (Support preservation on boundary-supported inputs)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, let $\alpha \in (0,1)$, and set $a := t - h^\alpha$. If $w \in \mathfrak{G}_t$ satisfies*

$$\mathrm{supp}(w) \subset [a,t],$$

then

$$\mathrm{supp}(\mathcal{S}_t w) \subset [a,t]. \tag{98}$$

Proof. Let $w \in \mathfrak{G}_t$ satisfy $\mathrm{supp}(w) \subset [a,t]$, and set $u := \mathcal{S}_t w$. Then

$$\mathcal{K}_{Y,t}u = \mathcal{K}_{XY,t}w.$$

Because w is supported in $[a,t]$, the Volterra form of $\mathcal{K}_{XY,t}$ yields

$$(\mathcal{K}_{XY,t}w)(r) = 0 \qquad \text{for } 0 \le r < a.$$

Hence

$$(\mathcal{K}_{Y,t}u)(r) = 0 \qquad \text{for } 0 \le r < a.$$

Equivalently,

$$\int_0^r K_Y(r,s)u(s)\,ds = 0 \qquad \text{for } 0 \le r < a.$$

This means exactly that the truncated operator $\mathcal{K}_{Y,a}$ annihilates $u|_{[0,a]}$. By Assumption 4.1, $\mathcal{K}_{Y,a}$ is injective, so $u|_{[0,a]} = 0$ almost everywhere. Therefore (98) holds. □

Definition 11.4 (Frozen right-sided Volterra model on the unit interval)**.** For $H \in (0, \frac{1}{2})$, define

$$(\mathfrak{I}_H\phi)(u) := \int_u^1 (v-u)^{H-\frac{1}{2}}\phi(v)\,dv, \qquad 0 \le u \le 1, \tag{99}$$

for $\phi \in E = L^2([0,1])$.

Definition 11.5 (Localized exact Volterra operators)**.** Fix $t \in (0,T)$, $\alpha \in (0,1)$, $h \in (0,t]$, and $i \in \{Y, XY\}$. Define the operator

$$\mathfrak{A}_{i,t,h,\alpha} : E \to E$$

by

$$(\mathfrak{A}_{i,t,h,\alpha}\phi)(u) := \int_u^1 (v-u)^{H_i-\frac{1}{2}}L_i(t-h^\alpha u, t-h^\alpha v)\phi(v)\,dv, \qquad 0 \le u \le 1. \tag{100}$$

Proposition 11.6 (Exact localized representation and frozen-kernel limit)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, $\alpha \in (0,1)$, and let $i \in \{Y, XY\}$. Then:*

(i) for every $\phi \in E$,

$$\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{i,t}\mathcal{L}^{(\alpha)}_{t,h}\phi = h^{\alpha H_i}\mathfrak{A}_{i,t,h,\alpha}\phi; \tag{101}$$

(ii) there exists $C < \infty$, depending only on t, α, i, such that

$$\|\mathfrak{A}_{i,t,h,\alpha} - L_i(t,t)\mathfrak{J}_{H_i}\|_{\mathcal{L}(E)} \le Ch^\alpha \tag{102}$$

for all sufficiently small $h > 0$.

Proof. Fix $\phi \in E$ and write $\mathcal{L} = \mathcal{L}^{(\alpha)}_{t,h}$, $\mathcal{D} = \mathcal{D}^{(\alpha)}_{t,h}$. For $0 \le u \le 1$,

$$\begin{aligned}(\mathcal{D}\mathcal{K}_{i,t}\mathcal{L}\phi)(u) &= h^{\alpha/2}\int_0^{t-h^\alpha u} K_i(t-h^\alpha u, s)\, h^{-\alpha/2}\phi\Big(\frac{t-s}{h^\alpha}\Big)\,\mathbf{1}_{[t-h^\alpha,t]}(s)\,ds \\ &= h^{\alpha/2}\int_{t-h^\alpha}^{t-h^\alpha u}(t-h^\alpha u - s)^{H_i-\frac12}L_i(t-h^\alpha u, s)\,h^{-\alpha/2}\phi\Big(\frac{t-s}{h^\alpha}\Big)\,ds.\end{aligned}$$

With the change of variables $s = t - h^\alpha v$, this becomes

$$\begin{aligned}(\mathcal{D}\mathcal{K}_{i,t}\mathcal{L}\phi)(u) &= h^{\alpha H_i}\int_u^1 (v-u)^{H_i-\frac12}L_i(t-h^\alpha u, t-h^\alpha v)\phi(v)\,dv \\ &= h^{\alpha H_i}(\mathfrak{A}_{i,t,h,\alpha}\phi)(u),\end{aligned}$$

which proves (101).

To prove (102), note that the difference kernel is

$$R_{i,t,h,\alpha}(u,v) := (v-u)^{H_i-\frac12}\big(L_i(t-h^\alpha u, t-h^\alpha v) - L_i(t,t)\big)\mathbf{1}_{\{0\le u\le v\le 1\}}.$$

Because L_i is C^1 near the diagonal, there exists $C < \infty$ such that

$$|L_i(t-h^\alpha u, t-h^\alpha v) - L_i(t,t)| \le Ch^\alpha \qquad \text{for } 0 \le u \le v \le 1.$$

Hence

$$|R_{i,t,h,\alpha}(u,v)| \le Ch^\alpha(v-u)^{H_i-\frac12}\mathbf{1}_{\{0\le u\le v\le 1\}}.$$

Since

$$\int_0^1\int_0^v (v-u)^{2H_i-1}\,du\,dv < \infty$$

for $H_i > 0$, the Hilbert–Schmidt norm of the difference is $O(h^\alpha)$. This implies (102). □

Definition 11.7 (Exact local transfer family)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, and $h \in (0,t]$. Define

$$\mathfrak{G}^{\mathrm{near}}_{t,h,\alpha} := \mathcal{D}^{(\alpha)}_{t,h}\big(\mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}\big) \subset E.$$

The *exact local transfer operator* is

$$\mathfrak{T}_{t,h,\alpha} : \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha} \subset E \to E, \qquad \mathfrak{T}_{t,h,\alpha}\phi := h^{\alpha(H_Y - H_{XY})}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t\mathcal{L}^{(\alpha)}_{t,h}\phi. \tag{103}$$

Proposition 11.8 (Exact localized transfer identity)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, and let $h > 0$ be sufficiently small. Then:*

(i) $\mathfrak{T}_{t,h,\alpha}$ is a well-defined closed operator on E;

(ii) for every $\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$,

$$\mathfrak{A}_{Y,t,h,\alpha}(\mathfrak{T}_{t,h,\alpha}\phi) = \mathfrak{A}_{XY,t,h,\alpha}\phi. \tag{104}$$

Proof. Well-definedness follows from Proposition 11.3. If $\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$, then $w := \mathcal{L}^{(\alpha)}_{t,h}\phi \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}$, so $\mathcal{S}_t w \in H^{\mathrm{near}}_{t,h,\alpha}$ and $\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t w \in E$.

To prove closedness, let $\phi_n \to \phi$ in E and $\mathfrak{T}_{t,h,\alpha}\phi_n \to \psi$ in E, with $\phi_n \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$. Applying the isometric lifts gives

$$w_n := \mathcal{L}^{(\alpha)}_{t,h}\phi_n \to w := \mathcal{L}^{(\alpha)}_{t,h}\phi \qquad \text{in } H_t$$

and

$$h^{\alpha(H_Y-H_{XY})}\mathcal{S}_t w_n = \mathcal{L}^{(\alpha)}_{t,h}\mathfrak{T}_{t,h,\alpha}\phi_n \longrightarrow \mathcal{L}^{(\alpha)}_{t,h}\psi \qquad \text{in } H_t.$$

Since $\mathcal{S}_t$ is closed, $w \in \mathfrak{G}_t$ and $h^{\alpha(H_Y-H_{XY})}\mathcal{S}_t w = \mathcal{L}^{(\alpha)}_{t,h}\psi$. Since $w = \mathcal{L}^{(\alpha)}_{t,h}\phi$, one already has $w \in H^{\mathrm{near}}_{t,h,\alpha}$, so $\phi = \mathcal{D}^{(\alpha)}_{t,h} w \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$ and $\mathfrak{T}_{t,h,\alpha}\phi = \psi$. Thus $\mathfrak{T}_{t,h,\alpha}$ is closed.

Now let $\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$ and set $w := \mathcal{L}^{(\alpha)}_{t,h}\phi$. Set

$$\beta := H_Y - H_{XY}.$$

By definition,

$$\mathcal{K}_{Y,t}\mathcal{S}_t w = \mathcal{K}_{XY,t} w.$$

Applying Proposition 11.6 to both sides and using $H_Y - H_{XY} = \beta$ gives

$$\begin{aligned} h^{\alpha H_Y}\mathfrak{A}_{Y,t,h,\alpha}\big(\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t w\big) &= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\big(\mathcal{D}^{(\alpha)}_{t,h} w\big) \\ &= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\phi. \end{aligned}$$

Multiplying by $h^{\alpha\beta-\alpha H_Y} = h^{-\alpha H_{XY}}$ yields

$$\mathfrak{A}_{Y,t,h,\alpha}\big(h^{\alpha\beta}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t w\big) = \mathfrak{A}_{XY,t,h,\alpha}\phi,$$

which is exactly (104). □

Definition 11.9 (Finite-interval right fractional integral and derivative)**.** For $\lambda \in (0,1)$ and $\phi \in E = L^2([0,1])$, define

$$(\mathfrak{I}^{\lambda}_{-,1}\phi)(u) := \frac{1}{\Gamma(\lambda)}\int_u^1 (v-u)^{\lambda-1}\phi(v)\,dv, \qquad 0 \le u \le 1. \tag{105}$$

For $\beta \in (0,1)$ and $\phi \in C_c^\infty((0,1))$, define

$$(\mathfrak{D}^{\beta}_{-,1}\phi)(u) := -\frac{d}{du}\left[\frac{1}{\Gamma(1-\beta)}\int_u^1 (v-u)^{-\beta}\phi(v)\,dv\right], \qquad 0 < u < 1. \tag{106}$$

These are the right-sided Riemann–Liouville operators; see Samko, Kilbas, and Marichev [18, Ch. 2].

Lemma 11.10 (Unit-interval semigroup and injectivity)**.** *Let $\lambda, \mu \in (0,1)$ with $\lambda + \mu \le 1$.*

(i) For every $\phi \in E$,

$$\mathfrak{I}^{\lambda}_{-,1}\mathfrak{I}^{\mu}_{-,1}\phi = \mathfrak{I}^{\lambda+\mu}_{-,1}\phi. \tag{107}$$

(ii) For every $H \in (0, \frac{1}{2})$, the operator $\mathfrak{I}_H$ of Definition 11.4 satisfies

$$\mathfrak{I}_H = \Gamma\Big(H + \frac{1}{2}\Big)\,\mathfrak{I}^{H+\frac{1}{2}}_{-,1}, \tag{108}$$

and is injective on E.

Proof. Let $\phi \in E$. For almost every $u \in [0,1]$, Fubini's theorem gives

$$\begin{aligned} (\mathfrak{I}^{\lambda}_{-,1}\mathfrak{I}^{\mu}_{-,1}\phi)(u) &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_u^1\int_r^1 (r-u)^{\lambda-1}(v-r)^{\mu-1}\phi(v)\,dv\,dr \\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_u^1\left(\int_u^v (r-u)^{\lambda-1}(v-r)^{\mu-1}\,dr\right)\phi(v)\,dv. \end{aligned}$$

With $r = u + (v-u)\theta$, the inner integral becomes

$$(v-u)^{\lambda+\mu-1}\int_0^1 \theta^{\lambda-1}(1-\theta)^{\mu-1}\,d\theta = (v-u)^{\lambda+\mu-1}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Substituting this identity proves (107).

Identity (108) follows from the definitions with $\lambda = H + \frac{1}{2}$. To prove injectivity, let $\mathfrak{I}_H\phi = 0$. Set $\lambda := H + \frac{1}{2} \in (0,1)$. By (108),

$$\mathfrak{I}^{\lambda}_{-,1}\phi = 0.$$

Applying (107) with $\mu = 1-\lambda$ yields

$$\mathfrak{I}^{1-\lambda}_{-,1}\mathfrak{I}^{\lambda}_{-,1}\phi = \mathfrak{I}^{1}_{-,1}\phi = 0.$$

But

$$(\mathfrak{I}^{1}_{-,1}\phi)(u) = \int_u^1 \phi(v)\,dv,$$

so differentiating almost everywhere gives $\phi = 0$ almost everywhere. Thus $\mathfrak{I}_H$ is injective. □

Definition 11.11 (Frozen transfer operator on the unit interval)**.** Assume

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Fix $t \in (0,T)$. Define the domain

$$\mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t) := \{\phi \in E : \mathfrak{I}_{H_{XY}}\phi \in \mathrm{Ran}(\mathfrak{I}_{H_Y})\}, \tag{109}$$

and the frozen transfer operator

$$\mathfrak{T}^{\mathrm{fr}}_t : \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t) \subset E \to E, \qquad \mathfrak{T}^{\mathrm{fr}}_t\phi := \frac{L_{XY}(t,t)}{L_Y(t,t)}\mathfrak{I}^{-1}_{H_Y}\mathfrak{I}_{H_{XY}}\phi. \tag{110}$$

Proposition 11.12 (Finite-interval fractional representation of the frozen transfer)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Then:

(i) for every $\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$,

$$L_Y(t,t)\mathfrak{I}_{H_Y}(\mathfrak{T}^{\mathrm{fr}}_t\phi) = L_{XY}(t,t)\mathfrak{I}_{H_{XY}}\phi; \tag{111}$$

(ii) for every $\phi \in C^\infty_c((0,1))$,

$$\mathfrak{T}^{\mathrm{fr}}_t\phi = \mathbf{c}_t\,\mathfrak{D}^{\beta}_{-,1}\phi, \tag{112}$$

where $\mathbf{c}_t$ is the coefficient from (82);

(iii) if $\widetilde{\phi}$ denotes the zero extension of $\phi \in C^\infty_c((0,1))$ to $\mathbb{R}$, then

$$\mathfrak{D}^{\beta}_{-,1}\phi = (\mathfrak{D}^{\beta}_{-}\widetilde{\phi})\big|_{(0,1)}. \tag{113}$$

Proof. Part (i) is exactly the definition of $\mathfrak{T}^{\mathrm{fr}}_t$ and injectivity of $\mathfrak{I}_{H_Y}$ from Lemma 11.10.

For part (ii), let $\phi \in C^\infty_c((0,1))$ and set

$$\psi := \mathbf{c}_t\,\mathfrak{D}^{\beta}_{-,1}\phi.$$

Since ϕ is compactly supported away from the boundary, $\psi \in E$. Using (108), the semigroup identity (107), and the left-inverse property

$$\mathfrak{I}^{1-\beta}_{-,1}\mathfrak{I}^{\beta}_{-,1} = \mathfrak{I}^{1}_{-,1}, \qquad -\frac{d}{du}\mathfrak{I}^{1}_{-,1}f = f \quad \text{a.e.},$$

we obtain

$$\mathfrak{I}^{H_Y+\frac{1}{2}}_{-,1}\mathfrak{D}^{\beta}_{-,1}\phi = \mathfrak{I}^{H_{XY}+\frac{1}{2}}_{-,1}\phi.$$

Multiplying by the Gamma factors gives

$$\begin{aligned} L_Y(t,t)\mathfrak{I}_{H_Y}\psi &= L_Y(t,t)\mathbf{c}_t\Gamma\left(H_Y+\frac{1}{2}\right)\mathfrak{I}^{H_Y+\frac{1}{2}}_{-,1}\mathfrak{D}^{\beta}_{-,1}\phi \\ &= L_{XY}(t,t)\Gamma\left(H_{XY}+\frac{1}{2}\right)\mathfrak{I}^{H_{XY}+\frac{1}{2}}_{-,1}\phi \\ &= L_{XY}(t,t)\mathfrak{I}_{H_{XY}}\phi. \end{aligned}$$

By part (i) and injectivity of $\mathfrak{I}_{H_Y}$, this proves (112).

For part (iii), if $\widetilde{\phi}$ is the zero extension of ϕ, then for $u \in (0,1)$,

$$\frac{1}{\Gamma(1-\beta)}\int_u^\infty (v-u)^{-\beta}\widetilde{\phi}(v)\,dv = \frac{1}{\Gamma(1-\beta)}\int_u^1 (v-u)^{-\beta}\phi(v)\,dv.$$

Differentiating yields (113). □

Proposition 11.13 (Weak local convergence under a uniform local bound)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Let $\phi \in C_c^\infty((0,1))$. Assume that:

(i) $\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$ for all sufficiently small $h > 0$;

(ii) there exists $C_\phi < \infty$ such that

$$\|\mathfrak{T}_{t,h,\alpha}\phi\|_E \le C_\phi \tag{114}$$

for all sufficiently small $h > 0$.

Then

$$\mathfrak{T}_{t,h,\alpha}\phi \rightharpoonup \mathfrak{T}^{\mathrm{fr}}_t\phi \qquad \text{weakly in } E = L^2([0,1]) \tag{115}$$

as $h \downarrow 0$.

Proof. Let $h_n \downarrow 0$. By (114), the sequence $(\mathfrak{T}_{t,h_n,\alpha}\phi)$ is bounded in E, so after passing to a subsequence we may assume

$$\mathfrak{T}_{t,h_n,\alpha}\phi \rightharpoonup \psi \qquad \text{weakly in } E$$

for some $\psi \in E$.

By Proposition 11.8,

$$\mathfrak{A}_{Y,t,h_n,\alpha}(\mathfrak{T}_{t,h_n,\alpha}\phi) = \mathfrak{A}_{XY,t,h_n,\alpha}\phi.$$

The right-hand side converges strongly in E to

$$L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi$$

by Proposition 11.6. On the left-hand side, decompose

$$\begin{aligned}\mathfrak{A}_{Y,t,h_n,\alpha}(\mathfrak{T}_{t,h_n,\alpha}\phi) &= \big(\mathfrak{A}_{Y,t,h_n,\alpha} - L_Y(t,t)\mathfrak{J}_{H_Y}\big)(\mathfrak{T}_{t,h_n,\alpha}\phi)\\ &\quad + L_Y(t,t)\mathfrak{J}_{H_Y}(\mathfrak{T}_{t,h_n,\alpha}\phi).\end{aligned}$$

The first term tends to 0 strongly in E because (102) gives operator-norm convergence and the sequence $(\mathfrak{T}_{t,h_n,\alpha}\phi)$ is bounded in E. Therefore

$$L_Y(t,t)\mathfrak{J}_{H_Y}(\mathfrak{T}_{t,h_n,\alpha}\phi) \longrightarrow L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi \qquad \text{strongly in } E.$$

Since $\mathfrak{J}_{H_Y}$ is bounded on E, weak convergence of $\mathfrak{T}_{t,h_n,\alpha}\phi$ implies

$$\mathfrak{J}_{H_Y}(\mathfrak{T}_{t,h_n,\alpha}\phi) \rightharpoonup \mathfrak{J}_{H_Y}\psi \qquad \text{weakly in } E.$$

Because the same sequence converges strongly, its weak limit must equal its strong limit, so

$$L_Y(t,t)\mathfrak{J}_{H_Y}\psi = L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi.$$

By Proposition 11.12(i) and injectivity of $\mathfrak{J}_{H_Y}$, this implies

$$\psi = \mathfrak{T}^{\mathrm{fr}}_t\phi.$$

Thus every weakly convergent subsequence has the same weak limit, and (115) follows. $\square$

Definition 11.14 (Local and frozen graph norms)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, let $\alpha \in (0,1)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

For $h > 0$ sufficiently small and $\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$, define

$$\|\phi\|^2_{\mathfrak{G}^{\mathrm{loc}}_{t,h,\alpha}} := \|\phi\|^2_E + \frac{\eta^2}{\nu_Y^2}\|\mathfrak{T}_{t,h,\alpha}\phi\|^2_E. \tag{116}$$

For $\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$, define

$$\|\phi\|^2_{\mathfrak{G}^{\mathrm{fr}}_t} := \|\phi\|^2_E + \frac{\eta^2}{\nu_Y^2}\|\mathfrak{T}^{\mathrm{fr}}_t\phi\|^2_E. \tag{117}$$

Proposition 11.15 (Weak graph compactness of the exact local transfer family)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Let $h_n \downarrow 0$ and let

$$\phi_n \in \mathfrak{G}^{\mathrm{near}}_{t,h_n,\alpha}$$

be such that

$$\sup_{n\geq 1} \|\phi_n\|_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}} < \infty. \tag{118}$$

Then there exist a subsequence, still denoted (ϕ_n), and some

$$\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$$

such that

$$\phi_n \rightharpoonup \phi \qquad \textit{weakly in } E, \tag{119}$$

$$\mathfrak{T}_{t,h_n,\alpha}\phi_n \rightharpoonup \mathfrak{T}^{\mathrm{fr}}_t\phi \qquad \textit{weakly in } E. \tag{120}$$

Moreover,

$$\|\phi\|_{\mathfrak{G}^{\mathrm{fr}}_t} \leq \liminf_{n\to\infty} \|\phi_n\|_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}}. \tag{121}$$

Proof. By (118) and Definition 11.14, both sequences (ϕ_n) and $(\mathfrak{T}_{t,h_n,\alpha}\phi_n)$ are bounded in E. Since E is a Hilbert space, after passing to a subsequence we may assume

$$\phi_n \rightharpoonup \phi, \qquad \mathfrak{T}_{t,h_n,\alpha}\phi_n \rightharpoonup \psi \qquad \text{weakly in } E$$

for some $\phi, \psi \in E$.

By Proposition 11.8,

$$\mathfrak{A}_{Y,t,h_n,\alpha}(\mathfrak{T}_{t,h_n,\alpha}\phi_n) = \mathfrak{A}_{XY,t,h_n,\alpha}\phi_n.$$

We claim that

$$\mathfrak{A}_{XY,t,h_n,\alpha}\phi_n \rightharpoonup L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi \qquad \text{weakly in } E, \tag{122}$$

$$\mathfrak{A}_{Y,t,h_n,\alpha}(\mathfrak{T}_{t,h_n,\alpha}\phi_n) \rightharpoonup L_Y(t,t)\mathfrak{J}_{H_Y}\psi \qquad \text{weakly in } E. \tag{123}$$

The first convergence is a direct consequence of Proposition 11.6. The decomposition

$$\mathfrak{A}_{XY,t,h_n,\alpha} = \big(\mathfrak{A}_{XY,t,h_n,\alpha} - L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\big) + L_{XY}(t,t)\mathfrak{J}_{H_{XY}}$$

has an operator-norm error term tending to 0 by (102). Since (ϕ_n) is bounded in E, that error contributes strongly, hence weakly, to 0; the frozen term converges weakly to $L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi$ because $\mathfrak{J}_{H_{XY}}$ is bounded on E. This proves (122). The proof of (123) is identical.

Passing to the weak limit in the exact identity therefore gives

$$L_Y(t,t)\mathfrak{J}_{H_Y}\psi = L_{XY}(t,t)\mathfrak{J}_{H_{XY}}\phi.$$

Since $L_Y(t,t) > 0$ by Assumption 3.1, this shows that $\mathfrak{J}_{H_{XY}}\phi \in \mathrm{Ran}(\mathfrak{J}_{H_Y})$, hence $\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$ by Definition 11.11. By Proposition 11.12(i) and injectivity of $\mathfrak{J}_{H_Y}$,

$$\psi = \mathfrak{T}^{\mathrm{fr}}_t\phi.$$

This proves (119) and (120).

Finally, weak lower semicontinuity of the norm in E yields

$$\|\phi\|_E \leq \liminf_{n\to\infty} \|\phi_n\|_E, \qquad \|\mathfrak{T}^{\mathrm{fr}}_t\phi\|_E = \|\psi\|_E \leq \liminf_{n\to\infty} \|\mathfrak{T}_{t,h_n,\alpha}\phi_n\|_E.$$

Squaring both inequalities, adding them with weight η^2/ν_Y^2, and using

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

for nonnegative sequences gives

$$\|\phi\|^2_{\mathfrak{G}^{\mathrm{fr}}_t} \leq \liminf_{n\to\infty} \|\phi_n\|^2_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}}.$$

Taking square roots proves (121). ∎

Corollary 11.16 (Unit local graph balls converge to the frozen graph)**.** *Under the assumptions of Proposition 11.15, let $h_n \downarrow 0$ and let*

$$\phi_n \in \mathfrak{G}^{\mathrm{near}}_{t,h_n,\alpha}$$

satisfy

$$\|\phi_n\|_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}} \le 1 \qquad \text{for every } n.$$

Then some subsequence converges weakly in E to an element

$$\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$$

with

$$\|\phi\|_{\mathfrak{G}^{\mathrm{fr}}_t} \le 1.$$

Proof. Apply Proposition 11.15 to the sequence (ϕ_n). The unit bound gives weak compactness of the graph pairs in $E \times E$, and the liminf inequality in that proposition places the weak limit in the frozen graph with frozen graph norm at most one. □

Definition 11.17 (Local graph recovery core)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

A linear subspace

$$\mathcal{C}_{t,\alpha} \subset \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$$

is called a *local graph recovery core* if:

(i) $\mathcal{C}_{t,\alpha}$ is dense in $\mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$ with respect to the frozen graph norm $\|\cdot\|_{\mathfrak{G}^{\mathrm{fr}}_t}$;

(ii) for every $\phi \in \mathcal{C}_{t,\alpha}$, there exists $h_\phi \in (0,t]$ such that

$$\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha} \qquad \text{for all } 0 < h \le h_\phi;$$

(iii) for every $\phi \in \mathcal{C}_{t,\alpha}$,

$$\|\mathfrak{T}_{t,h,\alpha}\phi - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \longrightarrow 0 \qquad \text{as } h \downarrow 0. \tag{124}$$

Theorem 11.18 (Recovery sequences from a local graph recovery core)**.** *Assume the hypotheses of Proposition 11.15, and suppose that there exists a local graph recovery core $\mathcal{C}_{t,\alpha}$ in the sense of Definition 11.17. Let $h_n \downarrow 0$, and let*

$$\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t).$$

Then there exists a sequence

$$\phi_n \in \mathfrak{G}^{\mathrm{near}}_{t,h_n,\alpha}$$

such that

$$\|\phi_n - \phi\|_E \longrightarrow 0, \tag{125}$$

$$\|\mathfrak{T}_{t,h_n,\alpha}\phi_n - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \longrightarrow 0, \tag{126}$$

and consequently

$$\|\phi_n\|_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}} \longrightarrow \|\phi\|_{\mathfrak{G}^{\mathrm{fr}}_t}. \tag{127}$$

Proof. By Definition 11.17(i), choose a sequence

$$\psi_m \in \mathcal{C}_{t,\alpha}, \qquad m \ge 1,$$

such that

$$\|\psi_m - \phi\|_E + \|\mathfrak{T}^{\mathrm{fr}}_t\psi_m - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \le 2^{-m} \qquad \text{for every } m \ge 1. \tag{128}$$

For each m, Definition 11.17(ii)–(iii) yields a number $\eta_m \in (0,t]$ such that for every $0 < h \le \eta_m$,

$$\psi_m \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha} \qquad \text{and} \qquad \|\mathfrak{T}_{t,h,\alpha}\psi_m - \mathfrak{T}^{\mathrm{fr}}_t\psi_m\|_E \le 2^{-m}. \tag{129}$$

After shrinking if necessary, we may assume that (η_m) is nonincreasing.

For each n, define

$$m(n) := \max\{m \in \{1,\dots,n\} : h_n \le \eta_m\}.$$

This set is nonempty for all sufficiently large n because $h_n \downarrow 0$ and $\eta_1 > 0$. Moreover, $m(n) \to \infty$: for every fixed M, one has $h_n \le \eta_M$ for all sufficiently large n, hence $m(n) \ge M$ eventually. Set

$$\phi_n := \psi_{m(n)}$$

for all sufficiently large n, and define the remaining finitely many terms by $\phi_n := 0$. By construction, $\phi_n \in \mathfrak{G}^{\mathrm{near}}_{t,h_n,\alpha}$ for every n.

Using (128),

$$\|\phi_n - \phi\|_E = \|\psi_{m(n)} - \phi\|_E \le 2^{-m(n)} \longrightarrow 0,$$

which proves (125). Likewise, by (129) and (128),

$$\begin{aligned}\|\mathfrak{T}_{t,h_n,\alpha}\phi_n - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E &\le \|\mathfrak{T}_{t,h_n,\alpha}\psi_{m(n)} - \mathfrak{T}^{\mathrm{fr}}_t\psi_{m(n)}\|_E \\ &\quad + \|\mathfrak{T}^{\mathrm{fr}}_t\psi_{m(n)} - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \\ &\le 2^{-m(n)} + 2^{-m(n)} \longrightarrow 0,\end{aligned}$$

which proves (126).

Finally,

$$\|\phi_n\|_E \to \|\phi\|_E, \qquad \|\mathfrak{T}_{t,h_n,\alpha}\phi_n\|_E \to \|\mathfrak{T}^{\mathrm{fr}}_t\phi\|_E,$$

by (125) and (126). Substituting into Definition 11.14 and (117) proves (127). ☐

Corollary 11.19 (Sequential Mosco convergence of the local transfer graphs)**.** *Assume the hypotheses of Theorem 11.18. For $h > 0$, define the graph*

$$\Gamma^{\mathrm{loc}}_{t,h,\alpha} := \left\{(\phi, \mathfrak{T}_{t,h,\alpha}\phi) : \phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}\right\} \subset E \times E,$$

and define the frozen graph

$$\Gamma^{\mathrm{fr}}_t := \left\{(\phi, \mathfrak{T}^{\mathrm{fr}}_t\phi) : \phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)\right\}.$$

Then along every sequence $h_n \downarrow 0$, the graphs $\Gamma^{\mathrm{loc}}_{t,h_n,\alpha}$ converge to Γ^{fr}_t in the sequential Mosco sense [19]*:*

- *(i) if $(\phi_n, \mathfrak{T}_{t,h_n,\alpha}\phi_n) \in \Gamma^{\mathrm{loc}}_{t,h_n,\alpha}$ is bounded in $E \times E$, then some subsequence converges weakly in $E \times E$ to an element of Γ^{fr}_t;*
- *(ii) every $(\phi, \mathfrak{T}^{\mathrm{fr}}_t\phi) \in \Gamma^{\mathrm{fr}}_t$ admits a sequence $(\phi_n, \mathfrak{T}_{t,h_n,\alpha}\phi_n) \in \Gamma^{\mathrm{loc}}_{t,h_n,\alpha}$ converging strongly in $E \times E$ to $(\phi, \mathfrak{T}^{\mathrm{fr}}_t\phi)$.*

Proof. For item (i), boundedness in $E \times E$ implies boundedness of $\|\phi_n\|_{\mathfrak{G}^{\mathrm{loc}}_{t,h_n,\alpha}}$ by Definition 11.14. Proposition 11.15 therefore gives a subsequence and a limit $\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$ such that

$$\phi_n \rightharpoonup \phi, \qquad \mathfrak{T}_{t,h_n,\alpha}\phi_n \rightharpoonup \mathfrak{T}^{\mathrm{fr}}_t\phi$$

weakly in E. This is weak convergence in $E \times E$ toward an element of Γ^{fr}_t.

Item (ii) is exactly Theorem 11.18. ☐

12. Strong local convergence on the smooth core

The local graph limit must be populated by recovery sequences. For the exact Volterra operators, strong local convergence can be verified on the smooth core $C^\infty_c((0,1))$. This does not prove graph density of that core in the frozen graph norm. It proves the analytic part of the recovery-core hypothesis: exact local solvability and strong convergence for fixed smooth profiles.

Proposition 12.1 (Conjugated target-side operator)**.** *Assume Assumption 3.1, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and set*

$$\lambda := H_Y + \frac{1}{2} \in (0,1).$$

For $h > 0$, define

$$a_{Y,t,h,\alpha}(u,v) := L_Y(t - h^\alpha u, t - h^\alpha v), \qquad 0 \le u \le v \le 1.$$

Then there exists a bounded operator

$$\mathfrak{B}_{Y,t,h,\alpha} \in \mathcal{L}(E)$$

such that:

(i) for every $\psi \in E$, the function $\mathfrak{I}^{1-\lambda}_{-,1}(\mathfrak{A}_{Y,t,h,\alpha}\psi)$ is absolutely continuous on $[0,1]$, and

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{Y,t,h,\alpha}\psi) = \mathfrak{B}_{Y,t,h,\alpha}\psi; \tag{130}$$

(ii) there exists $C < \infty$, depending only on t, α, such that

$$\|\mathfrak{B}_{Y,t,h,\alpha} - L_Y(t,t)I_E\|_{\mathcal{L}(E)} \le Ch^{\alpha} \tag{131}$$

for all sufficiently small $h > 0$.

Proof. For $\psi \in E$, set

$$f_h := \mathfrak{A}_{Y,t,h,\alpha}\psi.$$

Using (105) with order $1-\lambda$, Fubini's theorem gives for almost every $u \in [0,1]$,

$$\begin{aligned}(\mathfrak{I}^{1-\lambda}_{-,1}f_h)(u) &= \frac{1}{\Gamma(1-\lambda)}\int_u^1\int_r^1 (r-u)^{-\lambda}(v-r)^{\lambda-1}a_{Y,t,h,\alpha}(r,v)\psi(v)\,dv\,dr\\ &= \int_u^1 F_{Y,t,h,\alpha}(u,v)\psi(v)\,dv,\end{aligned}$$

where

$$F_{Y,t,h,\alpha}(u,v) := \frac{1}{\Gamma(1-\lambda)}\int_0^1 s^{-\lambda}(1-s)^{\lambda-1}a_{Y,t,h,\alpha}(u+(v-u)s,v)\,ds.$$

Since L_Y is C^1, the function $F_{Y,t,h,\alpha}$ is continuous on Δ_1, and

$$\partial_1 F_{Y,t,h,\alpha}(u,v) = \frac{1}{\Gamma(1-\lambda)}\int_0^1 s^{-\lambda}(1-s)^{\lambda}\partial_1 a_{Y,t,h,\alpha}(u+(v-u)s,v)\,ds$$

exists and is bounded. Therefore $u \mapsto (\mathfrak{I}^{1-\lambda}_{-,1}f_h)(u)$ is absolutely continuous and differentiation under the integral sign yields

$$\mathfrak{D}^{\lambda}_{-,1}f_h(u) = \Gamma(\lambda)a_{Y,t,h,\alpha}(u,u)\psi(u) - \int_u^1 G_{Y,t,h,\alpha}(u,v)\psi(v)\,dv$$

for almost every $u \in [0,1]$, where

$$G_{Y,t,h,\alpha}(u,v) := \frac{\Gamma(\lambda)}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 s^{-\lambda}(1-s)^{\lambda}\partial_1 a_{Y,t,h,\alpha}(u+(v-u)s,v)\,ds.$$

Define

$$\begin{aligned}(\mathfrak{B}_{Y,t,h,\alpha}\psi)(u) &:= a_{Y,t,h,\alpha}(u,u)\psi(u)\\ &\quad - \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_u^1\int_0^1 s^{-\lambda}(1-s)^{\lambda}\\ &\qquad\qquad \times \partial_1 a_{Y,t,h,\alpha}(u+(v-u)s,v)\,ds\,\psi(v)\,dv.\end{aligned}$$

This proves (130).

To prove (131), note first that

$$|a_{Y,t,h,\alpha}(u,u) - L_Y(t,t)| \le Ch^{\alpha} \qquad \text{for } 0 \le u \le 1,$$

because L_Y is C^1. Also,

$$\partial_1 a_{Y,t,h,\alpha}(x,v) = -h^{\alpha}\partial_1 L_Y(t-h^{\alpha}x, t-h^{\alpha}v),$$

so $|\partial_1 a_{Y,t,h,\alpha}(x,v)| \le Ch^{\alpha}$ uniformly on Δ_1. Therefore $\mathfrak{B}_{Y,t,h,\alpha} - a_{Y,t,h,\alpha}(\cdot,\cdot)_{\mathrm{diag}}I_E$ has integral kernel bounded by $Ch^{\alpha}\mathbf{1}_{\{0\le u\le v\le 1\}}$. This kernel has Hilbert–Schmidt norm $O(h^{\alpha})$, which proves (131). □

Corollary 12.2 (Uniform invertibility of the conjugated target operator)**.** *Under the assumptions of Proposition 12.1, there exist $h_0 \in (0,t]$ and $C_t < \infty$ such that for every $0 < h \le h_0$,*

$$\mathfrak{B}_{Y,t,h,\alpha} : E \to E$$

is invertible and

$$\|\mathfrak{B}^{-1}_{Y,t,h,\alpha}\|_{\mathcal{L}(E)} \le C_t. \tag{132}$$

Proof. By Assumption 3.1, $L_Y(t,t) > 0$. Proposition 12.1 gives

$$\left\| \frac{1}{L_Y(t,t)} \mathfrak{B}_{Y,t,h,\alpha} - I_E \right\|_{\mathcal{L}(E)} \le \frac{C}{L_Y(t,t)} h^\alpha.$$

For h small enough the right-hand side is $< 1/2$, so the Neumann series gives invertibility and (132). $\square$

Proposition 12.3 (Conjugated source asymptotics on the smooth core)**.** *Assume Assumption 3.1, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Let $\phi \in C_c^\infty((0,1))$. Then there exists $C_\phi < \infty$ such that for all sufficiently small $h > 0$,

$$\left\| \frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_{-,1} (\mathfrak{A}_{XY,t,h,\alpha} \phi) - L_Y(t,t) \mathfrak{T}_t^{\mathrm{fr}} \phi \right\|_E \le C_\phi h^\alpha. \tag{133}$$

Proof. Define

$$a_{XY,t,h,\alpha}(u,v) := L_{XY}(t - h^\alpha u, t - h^\alpha v), \qquad 0 \le u \le v \le 1.$$

Using (105) and the change of variables $r = u + (v-u)s$, one obtains for almost every $u \in [0,1]$,

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_{-,1} (\mathfrak{A}_{XY,t,h,\alpha} \phi)(u) = -\frac{d}{du} \int_u^1 (v-u)^{-\beta} g_{t,h,\alpha}(u,v) \phi(v)\, dv, \tag{134}$$

where

$$g_{t,h,\alpha}(u,v) := \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 s^{-\lambda} (1-s)^{\mu-1} a_{XY,t,h,\alpha}(u + (v-u)s, v)\, ds.$$

Because L_{XY} is C^1, one has

$$\| g_{t,h,\alpha} - g_t^{\mathrm{fr}} \|_{C^1(\Delta_1)} \le C h^\alpha, \tag{135}$$

where

$$g_t^{\mathrm{fr}} := \frac{L_{XY}(t,t)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(1-\beta)}.$$

Now write

$$R_{t,h,\alpha,\phi}(u) := \frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_{-,1} (\mathfrak{A}_{XY,t,h,\alpha} \phi)(u) - L_Y(t,t) \mathfrak{T}_t^{\mathrm{fr}} \phi(u).$$

Using (134), the constancy of g_t^{fr}, and Proposition 11.12(ii), this becomes

$$R_{t,h,\alpha,\phi}(u) = -\frac{d}{du} \int_u^1 (v-u)^{-\beta} \big(g_{t,h,\alpha}(u,v) - g_t^{\mathrm{fr}} \big) \phi(v)\, dv.$$

Since ϕ is compactly supported in $(0,1)$, the boundary term at $v = 1$ vanishes, and the substitution $v = u + r$ yields

$$\begin{aligned} R_{t,h,\alpha,\phi}(u) = -\int_0^{1-u} r^{-\beta} \Big[&(\partial_1 + \partial_2)\big(g_{t,h,\alpha} - g_t^{\mathrm{fr}}\big)(u, u+r)\, \phi(u+r) \\ &+ \big(g_{t,h,\alpha} - g_t^{\mathrm{fr}}\big)(u, u+r)\, \phi'(u+r) \Big]\, dr. \end{aligned}$$

By (135), the bracketed term is $O_\phi(h^\alpha)$ uniformly in u and r. Because $\beta \in (0,1)$,

$$\int_0^{1-u} r^{-\beta}\, dr \le \frac{1}{1-\beta} \qquad \text{for } 0 \le u \le 1.$$

Hence $\|R_{t,h,\alpha,\phi}\|_{L^\infty([0,1])} \le C_\phi h^\alpha$, which implies (133). $\square$

Theorem 12.4 (Strong local convergence on the smooth core)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then for every $\phi \in C_c^\infty((0,1))$ there exist $h_\phi \in (0,t]$ and $C_\phi < \infty$ such that for every $0 < h \le h_\phi$,

$$\phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$$

and

$$\|\mathfrak{T}_{t,h,\alpha}\phi - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \le C_\phi h^\alpha. \tag{136}$$

Proof. Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Fix $\phi \in C_c^\infty((0,1))$. For $h > 0$ sufficiently small, define

$$q_{t,h,\alpha,\phi} := \frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_{-,1}(\mathfrak{A}_{XY,t,h,\alpha}\phi) \in E.$$

By Corollary 12.2, the operator $\mathfrak{B}_{Y,t,h,\alpha}$ is invertible for small h, so we may set

$$\psi_h := \mathfrak{B}^{-1}_{Y,t,h,\alpha} q_{t,h,\alpha,\phi} \in E.$$

Using Proposition 12.3, Proposition 12.1, and (132),

$$\begin{aligned}\|\psi_h - \mathfrak{T}^{\mathrm{fr}}_t\phi\|_E &\le \|\mathfrak{B}^{-1}_{Y,t,h,\alpha}\|_{\mathcal{L}(E)}\Big(\|q_{t,h,\alpha,\phi} - L_Y(t,t)\mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \\ &\qquad + \|(L_Y(t,t)I_E - \mathfrak{B}_{Y,t,h,\alpha})\mathfrak{T}^{\mathrm{fr}}_t\phi\|_E\Big) \\ &\le C_\phi h^\alpha.\end{aligned}$$

Let

$$r_h := \mathfrak{A}_{Y,t,h,\alpha}\psi_h - \mathfrak{A}_{XY,t,h,\alpha}\phi.$$

By the definition of ψ_h, Proposition 12.1, and the definition of $q_{t,h,\alpha,\phi}$,

$$\mathfrak{D}^\lambda_{-,1} r_h = 0 \qquad \text{in } E.$$

Therefore $\mathfrak{I}^{1-\lambda}_{-,1} r_h$ is absolutely continuous with zero derivative, so it is constant. Since $(\mathfrak{I}^{1-\lambda}_{-,1} r_h)(1) = 0$, that constant is 0. By injectivity of $\mathfrak{I}^{1-\lambda}_{-,1}$, proved in Lemma 11.10,

$$r_h = 0.$$

Thus

$$\mathfrak{A}_{Y,t,h,\alpha}\psi_h = \mathfrak{A}_{XY,t,h,\alpha}\phi. \tag{137}$$

Now define

$$w_h := \mathcal{L}^{(\alpha)}_{t,h}\phi \in H^{\mathrm{near}}_{t,h,\alpha}, \qquad u_h := h^{-\alpha\beta}\mathcal{L}^{(\alpha)}_{t,h}\psi_h \in H^{\mathrm{near}}_{t,h,\alpha}.$$

By Proposition 11.6 and (137),

$$\begin{aligned}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}u_h &= h^{-\alpha\beta}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}\mathcal{L}^{(\alpha)}_{t,h}\psi_h \\ &= h^{-\alpha\beta+\alpha H_Y}\mathfrak{A}_{Y,t,h,\alpha}\psi_h \\ &= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\phi \\ &= \mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{XY,t}w_h.\end{aligned}$$

Both $\mathcal{K}_{Y,t}u_h$ and $\mathcal{K}_{XY,t}w_h$ are supported in $[t-h^\alpha, t]$, so Lemma 11.2 implies

$$\mathcal{K}_{Y,t}u_h = \mathcal{K}_{XY,t}w_h.$$

Hence $w_h \in \mathcal{D}(\mathcal{S}_t)$ and $\mathcal{S}_t w_h = u_h$. By Definition 11.7,

$$\phi = \mathcal{D}^{(\alpha)}_{t,h}w_h \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}$$

and

$$\mathfrak{T}_{t,h,\alpha}\phi = h^{\alpha\beta}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t w_h = h^{\alpha\beta}\mathcal{D}^{(\alpha)}_{t,h}u_h = \psi_h.$$

Combining this identity with the estimate on $\psi_h - \mathfrak{T}_t^{\mathrm{fr}}\phi$ proves (136). $\square$

Corollary 12.5 (Smooth core as a recovery core)**.** *Under the assumptions of Theorem 12.4, the space $C_c^\infty((0,1))$ satisfies items (ii) and (iii) of Definition 11.17. In particular, if $C_c^\infty((0,1))$ is dense in $\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ with respect to $\|\cdot\|_{\mathfrak{G}_t^{\mathrm{fr}}}$, then it is a local graph recovery core.*

Proof. Theorem 12.4 gives items (ii) and (iii) directly. If $C_c^\infty((0,1))$ is also graph dense, then item (i) of Definition 11.17 holds as well. $\square$

Definition 12.6 (Intrinsic exact local graph on the unit interval)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, and choose $h > 0$ sufficiently small that Corollary 12.2 applies. Define

$$\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}} := \{(\phi,\psi) \in E \times E : \mathfrak{A}_{Y,t,h,\alpha}\psi = \mathfrak{A}_{XY,t,h,\alpha}\phi\}.$$

Proposition 12.7 (Closed exact local solver on the unit interval)**.** *Assume the hypotheses of Definition 12.6. Then:*

(i) $\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$ *is a closed linear subspace of $E \times E$;*
(ii) $\mathfrak{A}_{Y,t,h,\alpha} : E \to E$ *is injective;*
(iii) there exists a unique closed linear operator

$$\mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}} : \mathcal{D}(\mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}}) \subset E \to E$$

whose graph is $\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$.

Proof. Since $\mathfrak{A}_{Y,t,h,\alpha}$ and $\mathfrak{A}_{XY,t,h,\alpha}$ are bounded on E, the map

$$(\phi,\psi) \longmapsto \mathfrak{A}_{Y,t,h,\alpha}\psi - \mathfrak{A}_{XY,t,h,\alpha}\phi$$

is bounded from $E \times E$ to E. Therefore $\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$, being its kernel, is a closed linear subspace. This proves (i).

To prove injectivity, let $\psi \in E$ satisfy

$$\mathfrak{A}_{Y,t,h,\alpha}\psi = 0.$$

Proposition 12.1 gives

$$\mathfrak{B}_{Y,t,h,\alpha}\psi = \frac{1}{\Gamma(\lambda)}\mathfrak{D}_{-,1}^{\lambda}(\mathfrak{A}_{Y,t,h,\alpha}\psi) = 0, \qquad \lambda := H_Y + \frac{1}{2}.$$

By Corollary 12.2, $\mathfrak{B}_{Y,t,h,\alpha}$ is invertible, so $\psi = 0$. This proves (ii).

Part (iii) now follows exactly as in Proposition 18.2: by (ii), for each $\phi \in E$ there is at most one $\psi \in E$ such that $(\phi,\psi) \in \Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$. Hence the closed linear subspace $\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$ is the graph of a unique closed linear operator. $\square$

Proposition 12.8 (Intrinsic and transfer-defined local graphs coincide)**.** *Assume the hypotheses of Definition 12.6. Then*

$$\mathcal{D}(\mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}}) = \mathfrak{G}_{t,h,\alpha}^{\mathrm{near}}, \qquad \mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}}\phi = \mathfrak{T}_{t,h,\alpha}\phi \quad \textit{for every } \phi \in \mathfrak{G}_{t,h,\alpha}^{\mathrm{near}}. \tag{138}$$

In particular,

$$\Gamma_{t,h,\alpha}^{\mathrm{loc,ex}} = \left\{(\phi, \mathfrak{T}_{t,h,\alpha}\phi) : \phi \in \mathfrak{G}_{t,h,\alpha}^{\mathrm{near}}\right\}.$$

Proof. Let $\phi \in \mathfrak{G}_{t,h,\alpha}^{\mathrm{near}}$, and set

$$\psi := \mathfrak{T}_{t,h,\alpha}\phi.$$

Proposition 11.8(ii) gives

$$\mathfrak{A}_{Y,t,h,\alpha}\psi = \mathfrak{A}_{XY,t,h,\alpha}\phi,$$

so $(\phi,\psi) \in \Gamma_{t,h,\alpha}^{\mathrm{loc,ex}}$. Hence

$$\mathfrak{G}_{t,h,\alpha}^{\mathrm{near}} \subset \mathcal{D}(\mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}})$$

and

$$\mathfrak{T}_{t,h,\alpha}^{\mathrm{loc,ex}}\phi = \mathfrak{T}_{t,h,\alpha}\phi \qquad \text{for every } \phi \in \mathfrak{G}_{t,h,\alpha}^{\mathrm{near}}.$$

Conversely, let
$$\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{loc,ex}}_{t,h,\alpha})$$
and set
$$\psi := \mathfrak{T}^{\mathrm{loc,ex}}_{t,h,\alpha}\phi.$$
Then
$$\mathfrak{A}_{Y,t,h,\alpha}\psi = \mathfrak{A}_{XY,t,h,\alpha}\phi.$$
Define
$$w := \mathcal{L}^{(\alpha)}_{t,h}\phi \in H^{\mathrm{near}}_{t,h,\alpha}, \qquad u := h^{-\alpha(H_Y - H_{XY})}\mathcal{L}^{(\alpha)}_{t,h}\psi \in H^{\mathrm{near}}_{t,h,\alpha}.$$
Applying Proposition 11.6 to both sides of the exact local identity gives
$$\begin{aligned}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}u &= h^{-\alpha(H_Y - H_{XY})}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}\mathcal{L}^{(\alpha)}_{t,h}\psi \\ &= h^{-\alpha(H_Y - H_{XY})+\alpha H_Y}\mathfrak{A}_{Y,t,h,\alpha}\psi \\ &= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\phi \\ &= \mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{XY,t}w.\end{aligned}$$
Both $\mathcal{K}_{Y,t}u$ and $\mathcal{K}_{XY,t}w$ are supported in $[t-h^\alpha, t]$, so Lemma 11.2 implies
$$\mathcal{K}_{Y,t}u = \mathcal{K}_{XY,t}w.$$
Hence $w \in \mathcal{D}(\mathcal{S}_t)$ and $\mathcal{S}_t w = u$. By Definition 11.7,
$$\phi = \mathcal{D}^{(\alpha)}_{t,h}w \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha},$$
and
$$\mathfrak{T}_{t,h,\alpha}\phi = h^{\alpha(H_Y - H_{XY})}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t w = h^{\alpha(H_Y - H_{XY})}\mathcal{D}^{(\alpha)}_{t,h}u = \psi.$$
This proves (138). □

Proposition 12.9 (Conjugated local commutator operators on the unit interval)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*
$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$
Let
$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1,$$
and define
$$\delta_\chi(u,v) := \begin{cases} \dfrac{\chi(v)-\chi(u)}{v-u}, & 0 \le u < v \le 1, \\ \chi'(u), & 0 \le u = v \le 1.\end{cases}$$
For $\Theta \in E$, define
$$(\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta)(u) := \int_u^1 (v-u)^{\lambda-1}(\chi(v)-\chi(u))L_Y(t-h^\alpha u, t-h^\alpha v)\Theta(v)\,dv, \tag{139}$$
$$(\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta)(u) := \int_u^1 (v-u)^{\mu-1}(\chi(v)-\chi(u))L_{XY}(t-h^\alpha u, t-h^\alpha v)\Theta(v)\,dv, \tag{140}$$
where
$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}.$$
Then there exist bounded operators
$$\mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}, \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \in \mathcal{L}(E)$$
such that for every $\Theta \in E$,
$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta) = \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta \qquad \text{in } E, \tag{141}$$
$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta) = \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta \qquad \text{in } E. \tag{142}$$

Moreover, there exists $C_t < \infty$ such that

$$\left\|\mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\right\|_{\mathcal{L}(E)} + \left\|\mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\right\|_{\mathcal{L}(E)} \le C_t\|\delta_\chi\|_{C^1(\Delta_1)}, \qquad \Delta_1 := \{0 \le u \le v \le 1\}, \tag{143}$$

for all sufficiently small $h > 0$.

Proof. For the target side, write

$$(\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta)(u) = \int_u^1 (v-u)^\lambda \delta_\chi(u,v) L_Y(t-h^\alpha u, t-h^\alpha v)\Theta(v)\,dv.$$

Exactly as in the proof of Proposition 19.8, Fubini's theorem gives

$$\mathfrak{I}^{1-\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta)(u) = \int_u^1 (v-u) F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,v)\Theta(v)\,dv,$$

where $F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}$ is C^1 on Δ_1. Differentiating under the integral sign yields

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\Theta)(u) = F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,u)\Theta(u) - \int_u^1 K^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,v)\Theta(v)\,dv$$

for almost every $u \in [0,1]$, where

$$K^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,v) := F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,v) + (v-u)\partial_1 F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}(u,v).$$

This defines the bounded operator $\mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}$.

Because L_Y is C^1 and $\delta_\chi \in C^1(\Delta_1)$, the same Beta-integral formula shows

$$\|F^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\|_{C^1(\Delta_1)} \le C_t\|\delta_\chi\|_{C^1(\Delta_1)}$$

uniformly for small h. Since Δ_1 has finite measure, the corresponding multiplication term and integral term are bounded on E, with norm bounded by $C_t\|\delta_\chi\|_{C^1(\Delta_1)}$.

For the source side, write

$$(\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta)(u) = \int_u^1 (v-u)^\mu \delta_\chi(u,v) L_{XY}(t-h^\alpha u, t-h^\alpha v)\Theta(v)\,dv.$$

Exactly as in Proposition 19.8, one obtains

$$\mathfrak{I}^{1-\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta)(u) = \int_u^1 (v-u)^{1-\beta} F^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}(u,v)\Theta(v)\,dv,$$

where $\beta := H_Y - H_{XY}$ and $F^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}$ is C^1 on Δ_1 with

$$\|F^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\|_{C^1(\Delta_1)} \le C_t\|\delta_\chi\|_{C^1(\Delta_1)}.$$

Since $1-\beta > 0$, differentiation under the integral sign gives

$$\begin{aligned}\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\Theta)(u) &= -(1-\beta)\int_u^1 (v-u)^{-\beta} F^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}(u,v)\Theta(v)\,dv \\ &\quad - \int_u^1 (v-u)^{1-\beta}\partial_1 F^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}(u,v)\Theta(v)\,dv.\end{aligned}$$

Because $\beta < \frac{1}{2}$, the kernels

$$(v-u)^{-\beta}\mathbf{1}_{\Delta_1}(u,v), \qquad (v-u)^{1-\beta}\mathbf{1}_{\Delta_1}(u,v)$$

are Hilbert–Schmidt on Δ_1. Hence the right-hand side defines a bounded operator $\mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \in \mathcal{L}(E)$ with norm bounded by $C_t\|\delta_\chi\|_{C^1(\Delta_1)}$. This proves (142) and (143). □

Theorem 12.10 (Smooth localization of an exact local pair)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Let $h > 0$ be sufficiently small that Corollary 12.2 applies, let

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1,$$

and let

$$(\phi, \psi) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}.$$

Define the conjugated local defect

$$\mathfrak{F}^{\mathrm{loc}}_{t,h,\alpha,\chi}(\phi,\psi) := \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi - \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi \in E, \tag{144}$$

and the corrected output

$$\psi^{\mathrm{loc}}_{\chi} := \chi\psi - \mathfrak{B}^{-1}_{Y,t,h,\alpha}\mathfrak{F}^{\mathrm{loc}}_{t,h,\alpha,\chi}(\phi,\psi). \tag{145}$$

Then:

(i)

$$(\chi\phi, \psi^{\mathrm{loc}}_{\chi}) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha};$$

(ii) there exists $C_t < \infty$ such that

$$\|\psi^{\mathrm{loc}}_{\chi} - \chi\psi\|_E \le C_t\|\delta_\chi\|_{C^1(\Delta_1)}(\|\phi\|_E + \|\psi\|_E); \tag{146}$$

(iii) consequently,

$$\|\chi\phi\|_E + \frac{\eta}{\nu_Y}\|\psi^{\mathrm{loc}}_{\chi}\|_E \le C_t(1 + \|\delta_\chi\|_{C^1(\Delta_1)})\left(\|\phi\|_E + \frac{\eta}{\nu_Y}\|\psi\|_E\right). \tag{147}$$

Proof. Because $(\phi, \psi) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}$, one has

$$\mathfrak{A}_{Y,t,h,\alpha}\psi = \mathfrak{A}_{XY,t,h,\alpha}\phi.$$

Multiplying by $\chi(u)$ and subtracting gives, for almost every $u \in [0,1]$,

$$\mathfrak{A}_{Y,t,h,\alpha}(\chi\psi)(u) - \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi)(u) = \int_u^1 K_Y(u,v)\,dv - \int_u^1 K_{XY}(u,v)\,dv,$$

where

$$K_Y(u,v) := (v-u)^{\lambda-1}(\chi(v) - \chi(u))L_Y(t - h^\alpha u, t - h^\alpha v)\psi(v),$$
$$K_{XY}(u,v) := (v-u)^{\mu-1}(\chi(v) - \chi(u))L_{XY}(t - h^\alpha u, t - h^\alpha v)\phi(v).$$

By the definition of the local commutator operators, this identity can be rewritten as

$$\mathfrak{A}_{Y,t,h,\alpha}(\chi\psi) - \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi) = \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi - \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi.$$

Applying Proposition 12.9 and then Proposition 12.1 yields

$$\mathfrak{B}_{Y,t,h,\alpha}(\chi\psi) = \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi)) + \mathfrak{F}^{\mathrm{loc}}_{t,h,\alpha,\chi}(\phi,\psi) \qquad \text{in } E.$$

By the definition of $\psi^{\mathrm{loc}}_{\chi}$,

$$\mathfrak{B}_{Y,t,h,\alpha}\psi^{\mathrm{loc}}_{\chi} = \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi)) \qquad \text{in } E.$$

Applying Proposition 12.1 again shows that

$$\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{Y,t,h,\alpha}\psi^{\mathrm{loc}}_{\chi} - \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi)) = 0 \qquad \text{in } E.$$

Exactly as in the proof of Theorem 12.4, injectivity of $\mathfrak{I}^{1-\lambda}_{-,1}$ implies

$$\mathfrak{A}_{Y,t,h,\alpha}\psi^{\mathrm{loc}}_{\chi} = \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi),$$

so $(\chi\phi, \psi^{\mathrm{loc}}_{\chi}) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}$. This proves (i).

For (ii), Corollary 12.2 and (143) give

$$\begin{aligned}\|\psi^{\mathrm{loc}}_{\chi} - \chi\psi\|_E &\le \|\mathfrak{B}^{-1}_{Y,t,h,\alpha}\|_{\mathcal{L}(E)}\left\|\mathfrak{F}^{\mathrm{loc}}_{t,h,\alpha,\chi}(\phi,\psi)\right\|_E \\ &\le C_t\|\delta_\chi\|_{C^1(\Delta_1)}(\|\phi\|_E + \|\psi\|_E),\end{aligned}$$

which proves (146).

Finally, $|\chi| \le 1$ and (146) imply

$$\begin{aligned}\|\chi\phi\|_E + \frac{\eta}{\nu_Y}\|\psi^{\mathrm{loc}}_\chi\|_E &\le \|\phi\|_E + \frac{\eta}{\nu_Y}\|\chi\psi\|_E + \frac{\eta}{\nu_Y}\|\psi^{\mathrm{loc}}_\chi - \chi\psi\|_E \\ &\le C_t(1 + \|\delta_\chi\|_{C^1(\Delta_1)})\left(\|\phi\|_E + \frac{\eta}{\nu_Y}\|\psi\|_E\right),\end{aligned}$$

which is (147). $\square$

Corollary 12.11 (Smooth cutoffs preserve boundary-supported local graph elements)**.** *Assume the hypotheses of Theorem 12.10. Let*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1,$$

and define

$$\chi_{t,h,\alpha}(s) := \chi\left(\frac{t-s}{h^\alpha}\right)\mathbf{1}_{[t-h^\alpha,t]}(s), \qquad 0 \le s \le t.$$

Then for every

$$w \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},$$

one has

$$\chi_{t,h,\alpha} w \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},$$

and there exists $C_t < \infty$, independent of w and h, such that

$$\|\chi_{t,h,\alpha} w\|_{H_t} + \frac{\eta}{\nu_Y} h^{\alpha(H_Y - H_{XY})}\|\mathcal{S}_t(\chi_{t,h,\alpha} w)\|_{H_t} \le C_t(1 + \|\delta_\chi\|_{C^1(\Delta_1)})\|w\|_{\mathfrak{G}_t}. \tag{148}$$

Proof. Set

$$\phi := \mathcal{D}^{(\alpha)}_{t,h} w \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}, \qquad \psi := \mathfrak{T}_{t,h,\alpha}\phi \in E.$$

By Proposition 12.8,

$$(\phi, \psi) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}.$$

Apply Theorem 12.10 to obtain

$$(\chi\phi, \psi^{\mathrm{loc}}_\chi) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}$$

and

$$\|\chi\phi\|_E + \frac{\eta}{\nu_Y}\|\psi^{\mathrm{loc}}_\chi\|_E \le C_t(1 + \|\delta_\chi\|_{C^1(\Delta_1)})\left(\|\phi\|_E + \frac{\eta}{\nu_Y}\|\psi\|_E\right).$$

Now define

$$\widetilde{w} := \mathcal{L}^{(\alpha)}_{t,h}(\chi\phi), \qquad \widetilde{u} := h^{-\alpha(H_Y - H_{XY})}\mathcal{L}^{(\alpha)}_{t,h}\psi^{\mathrm{loc}}_\chi.$$

By Proposition 12.8, the exact local pair relation implies

$$\widetilde{w} \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}, \qquad \mathcal{S}_t\widetilde{w} = \widetilde{u}.$$

Since

$$\widetilde{w}(s) = h^{-\alpha/2}\chi\left(\frac{t-s}{h^\alpha}\right)\phi\left(\frac{t-s}{h^\alpha}\right)\mathbf{1}_{[t-h^\alpha,t]}(s) = \chi_{t,h,\alpha}(s)w(s),$$

we have $\widetilde{w} = \chi_{t,h,\alpha} w$. Finally, Lemma 11.2 gives

$$\|\widetilde{w}\|_{H_t} = \|\chi\phi\|_E, \qquad \|\mathcal{S}_t\widetilde{w}\|_{H_t} = h^{-\alpha(H_Y - H_{XY})}\|\psi^{\mathrm{loc}}_\chi\|_E,$$

while

$$\|w\|_{H_t} = \|\phi\|_E, \qquad \|\mathcal{S}_t w\|_{H_t} = h^{-\alpha(H_Y - H_{XY})}\|\psi\|_E.$$

Therefore the preceding bound is exactly (148). $\square$

Proposition 12.12 (Slow variation of a fixed time-side cutoff on the micro scale)**.** *Fix $\alpha \in (0,1)$, let*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1,$$

and assume that for some $\delta_* \in (0,1)$,

$$\chi \equiv 1 \text{ on } [0, 1-\delta_*].$$

For $h \in (0,1]$, set

$$\varepsilon_{h,\alpha} := h^{1-\alpha}, \qquad \widetilde{\chi}_{h,\alpha}(r) := \chi(\varepsilon_{h,\alpha} r)\mathbf{1}_{[0,\varepsilon^{-1}_{h,\alpha}]}(r), \qquad r \ge 0.$$

Define, on

$$\Delta_{\varepsilon_{h,\alpha}^{-1}} := \{0 \le r \le q \le \varepsilon_{h,\alpha}^{-1}\},$$

the quotient kernel

$$\delta_{\widetilde{\chi}_{h,\alpha}}(r,q) := \begin{cases} \dfrac{\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)}{q-r}, & 0 \le r < q \le \varepsilon_{h,\alpha}^{-1}, \\ \widetilde{\chi}'_{h,\alpha}(r), & 0 \le r = q \le \varepsilon_{h,\alpha}^{-1}. \end{cases}$$

Then:

(i)

$$0 \le \widetilde{\chi}_{h,\alpha} \le 1, \qquad \widetilde{\chi}_{h,\alpha} \equiv 1 \text{ on } \left[0, (1-\delta_*)\varepsilon_{h,\alpha}^{-1}\right], \qquad \operatorname{supp} \widetilde{\chi}_{h,\alpha} \subset [0, \varepsilon_{h,\alpha}^{-1}];$$

(ii) there exists $C_\chi < \infty$, depending only on χ, such that

$$\|\delta_{\widetilde{\chi}_{h,\alpha}}\|_{C^1(\Delta_{\varepsilon_{h,\alpha}^{-1}})} \le C_\chi \varepsilon_{h,\alpha} \qquad \text{for every } h \in (0,1]. \tag{149}$$

Proof. Item (i) follows from the definition of $\widetilde{\chi}_{h,\alpha}$.

For $0 \le r < q \le \varepsilon_{h,\alpha}^{-1}$, set

$$\varepsilon := \varepsilon_{h,\alpha}.$$

Then

$$\delta_{\widetilde{\chi}_{h,\alpha}}(r,q) = \frac{\chi(\varepsilon q) - \chi(\varepsilon r)}{q-r} = \varepsilon \frac{\chi(\varepsilon q) - \chi(\varepsilon r)}{\varepsilon q - \varepsilon r}.$$

Hence, if

$$\delta_\chi(x,y) := \begin{cases} \dfrac{\chi(y) - \chi(x)}{y-x}, & 0 \le x < y \le 1, \\ \chi'(x), & 0 \le x = y \le 1, \end{cases}$$

then

$$\delta_{\widetilde{\chi}_{h,\alpha}}(r,q) = \varepsilon\, \delta_\chi(\varepsilon r, \varepsilon q) \qquad \text{on } \Delta_{\varepsilon^{-1}}.$$

Since $\chi \in C^\infty([0,1])$, one has $\delta_\chi \in C^1(\Delta_1)$. Therefore

$$\|\delta_{\widetilde{\chi}_{h,\alpha}}\|_{L^\infty(\Delta_{\varepsilon^{-1}})} \le \varepsilon \|\delta_\chi\|_{L^\infty(\Delta_1)}.$$

Differentiating the identity above yields

$$\partial_r \delta_{\widetilde{\chi}_{h,\alpha}}(r,q) = \varepsilon^2 (\partial_1 \delta_\chi)(\varepsilon r, \varepsilon q), \qquad \partial_q \delta_{\widetilde{\chi}_{h,\alpha}}(r,q) = \varepsilon^2 (\partial_2 \delta_\chi)(\varepsilon r, \varepsilon q),$$

and thus

$$\|\nabla \delta_{\widetilde{\chi}_{h,\alpha}}\|_{L^\infty(\Delta_{\varepsilon^{-1}})} \le \varepsilon^2 \|\nabla \delta_\chi\|_{L^\infty(\Delta_1)} \le \varepsilon \|\nabla \delta_\chi\|_{L^\infty(\Delta_1)},$$

because $\varepsilon \le 1$. This proves (149). □

Proposition 12.13 (Frozen domain as a fractional-integral range)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Then

$$\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}) = \mathfrak{I}_{-,1}^{\beta}(E). \tag{150}$$

The range identity has the following two forms.

(i) if $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$, then

$$\phi = \mathbf{c}_t^{-1} \mathfrak{I}_{-,1}^{\beta}(\mathfrak{T}_t^{\mathrm{fr}} \phi); \tag{151}$$

(ii) if $g \in E$ and

$$\phi := \mathbf{c}_t^{-1} \mathfrak{I}_{-,1}^{\beta} g,$$

then $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ and

$$\mathfrak{T}_t^{\mathrm{fr}} \phi = g. \tag{152}$$

Proof. Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2},$$

so that $\lambda = \mu + \beta$. Let $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$, and write

$$g := \mathfrak{T}_t^{\mathrm{fr}}\phi.$$

By Proposition 11.12(i) and Lemma 11.10,

$$\begin{aligned} L_Y(t,t)\Gamma(\lambda)\mathfrak{I}^{\lambda}_{-,1}g &= L_{XY}(t,t)\Gamma(\mu)\mathfrak{I}^{\mu}_{-,1}\phi \\ &= L_Y(t,t)\Gamma(\lambda)\mathbf{c}_t\,\mathfrak{I}^{\mu}_{-,1}\phi. \end{aligned}$$

Hence

$$\mathfrak{I}^{\mu}_{-,1}(\phi - \mathbf{c}_t^{-1}\mathfrak{I}^{\beta}_{-,1}g) = 0,$$

because $\mathfrak{I}^{\mu}_{-,1}\mathfrak{I}^{\beta}_{-,1} = \mathfrak{I}^{\lambda}_{-,1}$. Since $\mathfrak{I}^{\mu}_{-,1}$ is injective by the same argument as in Lemma 11.10, this proves (151).

Conversely, let $g \in E$ and set $\phi := \mathbf{c}_t^{-1}\mathfrak{I}^{\beta}_{-,1}g$. Then

$$\mathfrak{I}^{\mu}_{-,1}\phi = \mathbf{c}_t^{-1}\mathfrak{I}^{\lambda}_{-,1}g.$$

Multiplying by the Gamma factors built into $\mathbf{c}_t$ gives

$$L_{XY}(t,t)\Gamma(\mu)\mathfrak{I}^{\mu}_{-,1}\phi = L_Y(t,t)\Gamma(\lambda)\mathfrak{I}^{\lambda}_{-,1}g,$$

that is,

$$L_{XY}(t,t)\mathfrak{I}_{H_{XY}}\phi = L_Y(t,t)\mathfrak{I}_{H_Y}g.$$

Therefore $\mathfrak{I}_{H_{XY}}\phi \in \operatorname{Ran}(\mathfrak{I}_{H_Y})$, so $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ and (152) follows from Proposition 11.12(i) and injectivity of $\mathfrak{I}_{H_Y}$. □

Definition 12.14 (Finite-interval fractional Sobolev space)**.** Let $\beta \in (0, \frac{1}{2})$. The notation $H^{\beta}((0,1))$ denotes the Bessel potential space on the interval $(0,1)$, endowed with its Hilbert norm.

Theorem 12.15 (Density of the smooth core in the frozen graph)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Then:

(i) $\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}) = H^{\beta}((0,1))$ as sets;
(ii) the frozen graph norm is equivalent to the $H^{\beta}((0,1))$ norm;
(iii) $C_c^{\infty}((0,1))$ is dense in $\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ with respect to the frozen graph norm.

Proof. By Proposition 12.13,

$$\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}) = \mathfrak{I}^{\beta}_{-,1}(E).$$

For $\beta \in (0, \frac{1}{2})$, the classical finite-interval Riemann–Liouville isomorphism theorem identifies $\mathfrak{I}^{\beta}_{-,1}(E)$ with the right-sided fractional Sobolev space $H^{\beta}_0((0,1))$, and the inverse is the right-sided derivative $\mathfrak{D}^{\beta}_{-,1}$ in the L^2 sense. Since $\beta < \frac{1}{2}$, one has

$$H^{\beta}_0((0,1)) = H^{\beta}((0,1)),$$

so item (i) follows. This finite-interval fractional-Sobolev identification is used here as an analytic input; see [18, Ch. 13] and [20, Ch. 2].

Let $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$, and set $g := \mathfrak{T}_t^{\mathrm{fr}}\phi$. Proposition 12.13 gives

$$\phi = \mathbf{c}_t^{-1}\mathfrak{I}^{\beta}_{-,1}g.$$

Applying the inverse $\mathfrak{D}^{\beta}_{-,1}$ on the range $\mathfrak{I}^{\beta}_{-,1}(E)$ therefore yields

$$\mathfrak{D}^{\beta}_{-,1}\phi = \mathbf{c}_t^{-1}g.$$

By the same classical isomorphism, there exist constants $c_1, c_2 > 0$, depending only on β, such that

$$c_1(\|\phi\|_E + \|\mathfrak{D}^{\beta}_{-,1}\phi\|_E) \le \|\phi\|_{H^\beta((0,1))} \le c_2(\|\phi\|_E + \|\mathfrak{D}^{\beta}_{-,1}\phi\|_E).$$

Since $\mathfrak{D}^{\beta}_{-,1}\phi = \mathbf{c}_t^{-1}\mathfrak{T}^{\mathrm{fr}}_t\phi$, this is equivalent to

$$\|\phi\|_{H^\beta((0,1))} \asymp \|\phi\|_E + \|\mathfrak{T}^{\mathrm{fr}}_t\phi\|_E \asymp \|\phi\|_{\mathfrak{G}^{\mathrm{fr}}_t},$$

where the implicit constants depend only on β, η/ν_Y, and $|\mathbf{c}_t|$. This proves item (ii).

Finally, because $\beta < \frac{1}{2}$, the density theorem for Bessel potential spaces on bounded intervals yields

$$\overline{C_c^\infty((0,1))}^{H^\beta((0,1))} = H^\beta((0,1)).$$

See [20, Ch. 2]. By item (ii), this is equivalent to density in the frozen graph norm, proving item (iii). ☐

Corollary 12.16 (Smooth core as a local graph recovery core)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then $C_c^\infty((0,1))$ is a local graph recovery core in the sense of Definition 11.17.

Proof. By Theorem 12.15, $C_c^\infty((0,1))$ is dense in $\mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$ with respect to the frozen graph norm. By Theorem 12.4, it also satisfies items (ii) and (iii) of Definition 11.17. Hence it is a local graph recovery core. ☐

Corollary 12.17 (Smooth-core local recovery and Mosco convergence)**.** *Under the assumptions of Corollary 12.16, the conclusions of Theorem 11.18 and Corollary 11.19 hold without any additional recovery-core hypothesis.*

Proof. Apply Corollary 12.16 inside Theorem 11.18 and Corollary 11.19. ☐

13. Frozen buffered localization

The frozen right-sided fractional transfer has exact locality. Its action inside a boundary layer depends only on the input on a slightly larger boundary layer. On the frozen side this localization is exact, not asymptotic.

Proposition 13.1 (Global fractional representation of the frozen transfer)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Then for every

$$\phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t),$$

one has

$$\mathfrak{T}^{\mathrm{fr}}_t\phi = \mathbf{c}_t\,\mathfrak{D}^{\beta}_{-,1}\phi \qquad \textit{in } E. \tag{153}$$

Proof. Let

$$g := \mathfrak{T}^{\mathrm{fr}}_t\phi.$$

By Proposition 12.13,

$$\phi = \mathbf{c}_t^{-1}\mathfrak{I}^{\beta}_{-,1}g.$$

Applying $\mathfrak{D}^{\beta}_{-,1}$, which is the inverse of $\mathfrak{I}^{\beta}_{-,1}$ on its range by the finite-interval Riemann–Liouville isomorphism used in Theorem 12.15, gives

$$\mathfrak{D}^{\beta}_{-,1}\phi = \mathbf{c}_t^{-1}g.$$

Multiplying by $\mathbf{c}_t$ proves (153). ☐

Definition 13.2 (Buffered cutoff pair on the unit interval)**.** Let $0 < \rho_{\mathrm{in}} < \rho_{\mathrm{out}} < 1$. A pair

$$(\chi_{\mathrm{in}}, \chi_{\mathrm{out}}) \in C^\infty([0,1]) \times C^\infty([0,1])$$

is called a *buffered cutoff pair* with inner radius ρ_{in} and outer radius ρ_{out} if

$$0 \le \chi_{\mathrm{in}}, \chi_{\mathrm{out}} \le 1 \qquad \text{on } [0,1], \tag{154}$$

$$\operatorname{supp} \chi_{\mathrm{in}} \subset [1-\rho_{\mathrm{in}}, 1], \qquad \operatorname{supp} \chi_{\mathrm{out}} \subset [1-\rho_{\mathrm{out}}, 1], \tag{155}$$

$$\chi_{\mathrm{out}} \equiv 1 \qquad \text{on } [1-\rho_{\mathrm{in}}, 1]. \tag{156}$$

Proposition 13.3 (Smooth cutoffs preserve the frozen graph space)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Let $\chi \in C^\infty([0,1])$. Then multiplication by χ maps

$$\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$$

into itself, and there exists $C_\chi < \infty$ such that

$$\|\chi\phi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le C_\chi \|\phi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \qquad \textit{for every } \phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}). \tag{157}$$

Proof. By Theorem 12.15, one has

$$\mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}) = H^\beta((0,1)),$$

and the frozen graph norm is equivalent to the $H^\beta((0,1))$ norm. Since $\chi \in C^\infty([0,1])$, the smooth-multiplier theorem on Bessel potential spaces gives

$$\|\chi\phi\|_{H^\beta((0,1))} \le C'_\chi \|\phi\|_{H^\beta((0,1))} \qquad \text{for every } \phi \in H^\beta((0,1)).$$

See [20, Ch. 2]. Returning to the frozen graph norm via Theorem 12.15(ii) proves (157). □

Lemma 13.4 (Terminal vanishing propagates to the frozen transfer)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and fix $t \in (0,T)$. Let $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$, and suppose that for some $a \in (0,1)$,

$$\phi = 0 \qquad \textit{a.e. on } [a,1].$$

Then

$$\mathfrak{T}_t^{\mathrm{fr}}\phi = 0 \qquad \textit{a.e. on } [a,1]. \tag{158}$$

Proof. Set

$$g := \mathbf{c}_t^{-1} \mathfrak{T}_t^{\mathrm{fr}} \phi.$$

By Proposition 12.13,

$$\phi = \mathfrak{I}^\beta_{-,1} g.$$

Applying the semigroup identity gives

$$\mathfrak{I}^{1-\beta}_{-,1}\phi = \mathfrak{I}^1_{-,1} g.$$

Now fix $u \in [a,1]$. Since $\phi(v) = 0$ for almost every $v \in [u,1]$, the integral formula for $\mathfrak{I}^{1-\beta}_{-,1}\phi$ yields

$$(\mathfrak{I}^{1-\beta}_{-,1}\phi)(u) = 0.$$

Hence

$$(\mathfrak{I}^1_{-,1} g)(u) = \int_u^1 g(v)\, dv = 0 \qquad \text{for every } u \in [a,1].$$

The function $u \mapsto \mathfrak{I}^1_{-,1} g(u)$ is absolutely continuous, so differentiating almost everywhere on $[a,1]$ gives $g = 0$ almost everywhere on that interval. Multiplying by $\mathbf{c}_t$ proves (158). □

Theorem 13.5 (Exact buffered localization for the frozen transfer)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

fix $t \in (0,T)$, and let $(\chi_{\mathrm{in}}, \chi_{\mathrm{out}})$ be a buffered cutoff pair in the sense of Definition 13.2. Then:

(i) for every $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$,

$$\chi_{\mathrm{out}}\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}});$$

(ii) there exists $C_{\mathrm{out}} < \infty$ *such that*

$$\|\chi_{\mathrm{out}}\phi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le C_{\mathrm{out}}\|\phi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \qquad \text{for every } \phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}); \tag{159}$$

(iii) for every $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$,

$$\chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}\phi = \chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}(\chi_{\mathrm{out}}\phi) \qquad \text{in } E. \tag{160}$$

Proof. Parts (i) and (ii) are exactly Proposition 13.3 with $\chi = \chi_{\mathrm{out}}$.

For part (iii), set

$$\psi := (1 - \chi_{\mathrm{out}})\phi.$$

By (156), one has $\psi = 0$ on $[1 - \rho_{\mathrm{in}}, 1]$. Since $\chi_{\mathrm{out}}\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ by part (i), also

$$\psi = \phi - \chi_{\mathrm{out}}\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}).$$

Lemma 13.4 with $a = 1 - \rho_{\mathrm{in}}$ therefore gives

$$\mathfrak{T}_t^{\mathrm{fr}}\psi = 0 \qquad \text{a.e. on } [1 - \rho_{\mathrm{in}}, 1].$$

Because $\operatorname{supp}\chi_{\mathrm{in}} \subset [1 - \rho_{\mathrm{in}}, 1]$, this implies

$$\chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}\psi = 0.$$

Since

$$\psi = \phi - \chi_{\mathrm{out}}\phi,$$

we conclude that

$$\chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}\phi - \chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}(\chi_{\mathrm{out}}\phi) = \chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}\psi = 0,$$

which is exactly (160). ☐

Corollary 13.6 (Inner-supported frozen graph functionals reduce to buffered profiles)**.** *Assume the hypotheses of Theorem 13.5. Let* $a, b \in E$ *satisfy*

$$\operatorname{supp} a \subset \operatorname{supp}\chi_{\mathrm{in}}, \qquad \operatorname{supp} b \subset \operatorname{supp}\chi_{\mathrm{in}},$$

and define

$$\Lambda(\phi) := \langle a, \phi\rangle_E + \langle b, \mathfrak{T}_t^{\mathrm{fr}}\phi\rangle_E, \qquad \phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}).$$

Then Λ *is continuous on* $\mathfrak{G}_t^{\mathrm{fr}}$, *and for every* $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$,

$$\Lambda(\phi) = \Lambda(\chi_{\mathrm{out}}\phi). \tag{161}$$

Consequently,

$$\|\Lambda\|_{(\mathfrak{G}_t^{\mathrm{fr}})^*} \le C_{\mathrm{out}} \sup\left\{|\Lambda(\psi)| : \psi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}),\ \operatorname{supp}\psi \subset \operatorname{supp}\chi_{\mathrm{out}},\ \|\psi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le 1\right\}, \tag{162}$$

where C_{out} *is the constant from* (159).

Proof. Continuity follows from Cauchy–Schwarz and Definition 11.14. Since a and b are supported in $\operatorname{supp}\chi_{\mathrm{in}}$, one has

$$\langle a, \phi\rangle_E = \langle a, \chi_{\mathrm{out}}\phi\rangle_E,$$

because $\chi_{\mathrm{out}} \equiv 1$ on $\operatorname{supp}\chi_{\mathrm{in}}$, and Theorem 13.5(iii) gives

$$\chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}\phi = \chi_{\mathrm{in}}\mathfrak{T}_t^{\mathrm{fr}}(\chi_{\mathrm{out}}\phi).$$

Therefore

$$\langle b, \mathfrak{T}_t^{\mathrm{fr}}\phi\rangle_E = \langle b, \mathfrak{T}_t^{\mathrm{fr}}(\chi_{\mathrm{out}}\phi)\rangle_E,$$

because b is supported in $\operatorname{supp}\chi_{\mathrm{in}}$. Hence (161) holds. For $\phi \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}})$ with $\|\phi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le 1$, set $\psi := \chi_{\mathrm{out}}\phi$. Then $\operatorname{supp}\psi \subset \operatorname{supp}\chi_{\mathrm{out}}$, and by Theorem 13.5(ii),

$$\|\psi\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le C_{\mathrm{out}}.$$

Hence

$$|\Lambda(\phi)| = |\Lambda(\psi)| \le C_{\mathrm{out}} \sup\left\{|\Lambda(\vartheta)| : \vartheta \in \mathcal{D}(\mathfrak{T}_t^{\mathrm{fr}}),\ \operatorname{supp}\vartheta \subset \operatorname{supp}\chi_{\mathrm{out}},\ \|\vartheta\|_{\mathfrak{G}_t^{\mathrm{fr}}} \le 1\right\},$$

and taking the supremum over graph-unit ϕ proves (162). ☐

14. Two-scale factorization of the singular functional

The local transfer family on the unit interval captures graph geometry at the coarse scale h^α. The gain functional is more singular. After restriction to that boundary layer, the increment profiles still concentrate on the smaller micro-scale

$$\varepsilon_{h,\alpha} := h^{1-\alpha}$$

near the endpoint $u = 0$. A second dilation identifies the exact micro-local operator induced by the local transfer family and recovers the same universal half-line profiles as the original h-dilation.

Definition 14.1 (Micro-dilation and micro-lift on the unit interval)**.** For $\varepsilon \in (0, 1]$, define

$$\mathfrak{d}_\varepsilon : E \to L^2(\mathbb{R}_+), \qquad (\mathfrak{d}_\varepsilon \phi)(r) := \varepsilon^{1/2}\phi(\varepsilon r)\mathbf{1}_{[0,\varepsilon^{-1}]}(r),$$

for $\phi \in E = L^2([0, 1])$. Also define

$$\mathfrak{l}_\varepsilon : L^2(\mathbb{R}_+) \to E, \qquad (\mathfrak{l}_\varepsilon \Phi)(u) := \varepsilon^{-1/2}\Phi\left(\frac{u}{\varepsilon}\right), \qquad 0 \le u \le 1,$$

for $\Phi \in L^2(\mathbb{R}_+)$. Finally, let

$$\mathfrak{p}_\varepsilon : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \qquad (\mathfrak{p}_\varepsilon \Phi)(r) := \Phi(r)\mathbf{1}_{[0,\varepsilon^{-1}]}(r),$$

denote truncation to the interval $[0, \varepsilon^{-1}]$.

Lemma 14.2 (Micro-dilation isometry and truncation identities)**.** *Let $\varepsilon \in (0, 1]$. Then:*

(i) $\mathfrak{d}_\varepsilon : E \to L^2(\mathbb{R}_+)$ is a linear isometric embedding;

(ii) for every $\phi \in E$ and $\Phi \in L^2(\mathbb{R}_+)$,

$$\langle \mathfrak{d}_\varepsilon \phi, \Phi\rangle_{L^2(\mathbb{R}_+)} = \langle \phi, \mathfrak{l}_\varepsilon \Phi\rangle_E; \tag{163}$$

(iii)

$$\mathfrak{l}_\varepsilon \mathfrak{d}_\varepsilon = I_E, \qquad \mathfrak{d}_\varepsilon \mathfrak{l}_\varepsilon = \mathfrak{p}_\varepsilon. \tag{164}$$

Proof. For $\phi \in E$,

$$\|\mathfrak{d}_\varepsilon \phi\|^2_{L^2(\mathbb{R}_+)} = \int_0^{\varepsilon^{-1}} \varepsilon|\phi(\varepsilon r)|^2\,dr = \int_0^1 |\phi(u)|^2\,du = \|\phi\|^2_E,$$

which proves (i). Likewise, for $\phi \in E$ and $\Phi \in L^2(\mathbb{R}_+)$, the change of variables $u = \varepsilon r$ gives

$$\langle \mathfrak{d}_\varepsilon \phi, \Phi\rangle_{L^2(\mathbb{R}_+)} = \int_0^{\varepsilon^{-1}} \varepsilon^{1/2}\phi(\varepsilon r)\Phi(r)\,dr = \int_0^1 \phi(u)\varepsilon^{-1/2}\Phi\left(\frac{u}{\varepsilon}\right)\,du = \langle \phi, \mathfrak{l}_\varepsilon \Phi\rangle_E,$$

which is (163). Finally,

$$(\mathfrak{l}_\varepsilon \mathfrak{d}_\varepsilon \phi)(u) = \varepsilon^{-1/2}\varepsilon^{1/2}\phi(u) = \phi(u), \qquad 0 \le u \le 1,$$

while

$$(\mathfrak{d}_\varepsilon \mathfrak{l}_\varepsilon \Phi)(r) = \varepsilon^{1/2}\varepsilon^{-1/2}\Phi(r)\mathbf{1}_{[0,\varepsilon^{-1}]}(r) = (\mathfrak{p}_\varepsilon \Phi)(r).$$

This proves (164). □

Proposition 14.3 (Second dilation inside the h^α boundary layer)**.** *Fix $t \in (0, T)$, $\alpha \in (0, 1)$, and $h \in (0, t]$, and set*

$$\varepsilon_{h,\alpha} := h^{1-\alpha}.$$

Then:

(i) for every $f \in H^{\mathrm{near}}_{t,h,\alpha}$,

$$\mathcal{D}_{t,h} f = \mathfrak{d}_{\varepsilon_{h,\alpha}}(\mathcal{D}^{(\alpha)}_{t,h} f); \tag{165}$$

(ii) for every $\Phi \in L^2(\mathbb{R}_+)$,

$$\mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi = \mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi; \tag{166}$$

(iii) if $\Phi \in L^2(\mathbb{R}_+)$ satisfies

$$\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha},$$

then

$$\varepsilon^{\beta}_{h,\alpha}\mathfrak{d}_{\varepsilon_{h,\alpha}}\Big(\mathfrak{T}_{t,h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi\Big) = h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi, \tag{167}$$

where $\beta := H_Y - H_{XY}$.

Proof. For $f \in H^{\mathrm{near}}_{t,h,\alpha}$ and $r \ge 0$,

$$\begin{aligned}(\mathfrak{d}_{\varepsilon_{h,\alpha}}\mathcal{D}^{(\alpha)}_{t,h}f)(r) &= \varepsilon^{1/2}_{h,\alpha}h^{\alpha/2}f(t-h^{\alpha}\varepsilon_{h,\alpha}r)\mathbf{1}_{[0,\varepsilon^{-1}_{h,\alpha}]}(r)\\ &= h^{1/2}f(t-hr)\mathbf{1}_{[0,\varepsilon^{-1}_{h,\alpha}]}(r).\end{aligned}$$

Because f is supported in $[t-h^{\alpha},t]$, the right-hand side vanishes for $r > \varepsilon^{-1}_{h,\alpha} = h^{\alpha-1}$. Hence it equals $\mathcal{D}_{t,h}f(r)$, proving (165).

For $\Phi \in L^2(\mathbb{R}_+)$ and $0 \le s \le t$,

$$\begin{aligned}(\mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi)(s) &= h^{-\alpha/2}\varepsilon^{-1/2}_{h,\alpha}\Phi\left(\frac{t-s}{h^{\alpha}\varepsilon_{h,\alpha}}\right)\mathbf{1}_{[t-h^{\alpha},t]}(s)\\ &= h^{-1/2}\Phi\left(\frac{t-s}{h}\right)\mathbf{1}_{[t-h^{\alpha},t]}(s).\end{aligned}$$

On the other hand,

$$(\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi)(s) = h^{-1/2}\Phi\left(\frac{t-s}{h}\right)\mathbf{1}_{\{0\le(t-s)/h\le\varepsilon^{-1}_{h,\alpha}\}}(s)\mathbf{1}_{[0,t]}(s).$$

But

$$0 \le \frac{t-s}{h} \le \varepsilon^{-1}_{h,\alpha} \iff t-h^{\alpha} \le s \le t,$$

so the two expressions coincide. Therefore (166) holds.

For part (iii), Definition 11.7 gives

$$\mathfrak{T}_{t,h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi = h^{\alpha\beta}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{S}_t\mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi.$$

Applying $\varepsilon^{\beta}_{h,\alpha}\mathfrak{d}_{\varepsilon_{h,\alpha}}$ and using (165) and (166) yields

$$\begin{aligned}\varepsilon^{\beta}_{h,\alpha}\mathfrak{d}_{\varepsilon_{h,\alpha}}\Big(\mathfrak{T}_{t,h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi\Big) &= \varepsilon^{\beta}_{h,\alpha}h^{\alpha\beta}\mathcal{D}_{t,h}\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi\\ &= h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi,\end{aligned}$$

because $\varepsilon^{\beta}_{h,\alpha}h^{\alpha\beta} = h^{\beta}$. This proves (167). □

Definition 14.4 (Micro-dilated exact and frozen Volterra operators)**.** Fix $t \in (0,T)$, $\alpha \in (0,1)$, let $i \in \{Y, XY\}$, and let $h > 0$ be sufficiently small that $h^{\alpha} \le t$. Set

$$\varepsilon_{h,\alpha} := h^{1-\alpha}.$$

The *micro-dilated exact Volterra operator* is

$$\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+),$$

defined by

$$(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi)(r) := \int_r^{\varepsilon^{-1}_{h,\alpha}} (q-r)^{H_i-\frac{1}{2}}L_i(t-hr,t-hq)\Phi(q)\,dq, \qquad r \ge 0. \tag{168}$$

Also define the *frozen half-line Volterra operator*

$$\mathfrak{V}^{\mathrm{hl}}_{i,t} : C_c^{\infty}((0,\infty)) \to L^2(\mathbb{R}_+)$$

by

$$(\mathfrak{V}^{\mathrm{hl}}_{i,t}\Phi)(r) := L_i(t,t)\int_r^{\infty}(q-r)^{H_i-\frac{1}{2}}\Phi(q)\,dq, \qquad r \ge 0. \tag{169}$$

Proposition 14.5 (Second blow-up of the localized Volterra operators)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, $\alpha \in (0,1)$, and let $i \in \{Y, XY\}$. For every sufficiently small $h > 0$ and every $\Phi \in L^2(\mathbb{R}_+)$,*

$$\mathfrak{d}_{\varepsilon_{h,\alpha}} \mathfrak{A}_{i,t,h,\alpha} \mathfrak{l}_{\varepsilon_{h,\alpha}} \Phi = \varepsilon_{h,\alpha}^{H_i} \mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha} \Phi \qquad \text{in } L^2(\mathbb{R}_+). \tag{170}$$

In particular,

$$\operatorname{supp}(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi) \subset [0, \varepsilon_{h,\alpha}^{-1}] \qquad \text{for every } \Phi \in L^2(\mathbb{R}_+). \tag{171}$$

Proof. Fix $h > 0$ sufficiently small and abbreviate $\varepsilon := \varepsilon_{h,\alpha}$. For $r \geq 0$,

$$\begin{aligned}(\mathfrak{d}_\varepsilon \mathfrak{A}_{i,t,h,\alpha} \mathfrak{l}_\varepsilon \Phi)(r) &= \varepsilon^{1/2} (\mathfrak{A}_{i,t,h,\alpha} \mathfrak{l}_\varepsilon \Phi)(\varepsilon r) \mathbf{1}_{[0,\varepsilon^{-1}]}(r) \\ &= \varepsilon^{1/2} \int_{\varepsilon r}^1 (v - \varepsilon r)^{H_i - \frac{1}{2}} L_i(t - h^\alpha \varepsilon r, t - h^\alpha v)\, \varepsilon^{-1/2} \Phi(v/\varepsilon)\, dv\, \mathbf{1}_{[0,\varepsilon^{-1}]}(r).\end{aligned}$$

With the change of variables $v = \varepsilon q$, and using $h^\alpha \varepsilon = h$, this becomes

$$\begin{aligned}(\mathfrak{d}_\varepsilon \mathfrak{A}_{i,t,h,\alpha} \mathfrak{l}_\varepsilon \Phi)(r) &= \varepsilon^{H_i} \int_r^{\varepsilon^{-1}} (q - r)^{H_i - \frac{1}{2}} L_i(t - hr, t - hq) \Phi(q)\, dq\, \mathbf{1}_{[0,\varepsilon^{-1}]}(r) \\ &= \varepsilon^{H_i} (\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha} \Phi)(r),\end{aligned}$$

which proves (170). The support statement (171) follows from (168). □

Corollary 14.6 (Exact micro-transfer identity through the micro-Volterra pair)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, and let $\alpha \in (0,1)$. Then for every sufficiently small $h > 0$ and every*

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}),$$

one has

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi. \tag{172}$$

Proof. Set $\varepsilon := \varepsilon_{h,\alpha}$. By Definition 14.8,

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi = \varepsilon^\beta \mathfrak{d}_\varepsilon \mathfrak{T}_{t,h,\alpha} \mathfrak{l}_\varepsilon \Phi.$$

Hence, using $\mathfrak{l}_\varepsilon \mathfrak{d}_\varepsilon = I_E$ from Lemma 14.2,

$$\begin{aligned}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) &= \varepsilon^{-H_Y} \mathfrak{d}_\varepsilon \mathfrak{A}_{Y,t,h,\alpha} \mathfrak{l}_\varepsilon (\varepsilon^\beta \mathfrak{d}_\varepsilon \mathfrak{T}_{t,h,\alpha} \mathfrak{l}_\varepsilon \Phi) \\ &= \varepsilon^{-H_Y + \beta} \mathfrak{d}_\varepsilon \mathfrak{A}_{Y,t,h,\alpha} \mathfrak{T}_{t,h,\alpha} \mathfrak{l}_\varepsilon \Phi.\end{aligned}$$

By the exact local transfer identity (104),

$$\mathfrak{A}_{Y,t,h,\alpha} \mathfrak{T}_{t,h,\alpha} \mathfrak{l}_\varepsilon \Phi = \mathfrak{A}_{XY,t,h,\alpha} \mathfrak{l}_\varepsilon \Phi.$$

Since $\beta = H_Y - H_{XY}$, it follows that

$$\begin{aligned}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) &= \varepsilon^{-H_{XY}} \mathfrak{d}_\varepsilon \mathfrak{A}_{XY,t,h,\alpha} \mathfrak{l}_\varepsilon \Phi \\ &= \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi,\end{aligned}$$

where the last step is Proposition 14.5. □

Proposition 14.7 (Frozen half-line limit on compactly supported profiles)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, $\alpha \in (0,1)$, let $i \in \{Y, XY\}$, and let $\Phi \in C_c^\infty((0,\infty))$. Then*

$$\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi \longrightarrow \mathfrak{V}^{\mathrm{hl}}_{i,t}\Phi \qquad \text{in } L^2(\mathbb{R}_+) \tag{173}$$

as $h \downarrow 0$.

Proof. Choose $R > 0$ such that $\operatorname{supp}\Phi \subset [0,R]$. For h sufficiently small one has $\varepsilon_{h,\alpha}^{-1} > R$ and $hR < t$, so for $0 \leq r \leq R$,

$$(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi)(r) = \int_r^R (q - r)^{H_i - \frac{1}{2}} L_i(t - hr, t - hq) \Phi(q)\, dq,$$

while $\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi(r) = 0$ for $r > R$. Likewise,

$$(\mathfrak{V}^{\mathrm{hl}}_{i,t}\Phi)(r) = L_i(t,t)\int_r^R (q-r)^{H_i-\frac{1}{2}}\Phi(q)\,dq, \qquad r \geq 0.$$

Since L_i is continuous near (t,t), one has pointwise convergence

$$(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi)(r) \longrightarrow (\mathfrak{V}^{\mathrm{hl}}_{i,t}\Phi)(r) \qquad \text{for every } r \geq 0.$$

Moreover, for $0 \leq r \leq R$,

$$\left|(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha}\Phi)(r)\right| \leq C\int_r^R (q-r)^{H_i-\frac{1}{2}}|\Phi(q)|\,dq,$$

with C independent of h. The right-hand side belongs to $L^2([0,R])$, because the kernel $(q-r)^{H_i-\frac{1}{2}}\mathbf{1}_{\{0\leq r\leq q\leq R\}}$ is Hilbert–Schmidt. Therefore dominated convergence gives (173). $\square$

Definition 14.8 (Micro-dilated local profiles and transfer family)**.** Fix $t \in (0,T)$, $\alpha \in (0,1)$, and $h \in (0,t]$, and set

$$\varepsilon_{h,\alpha} := h^{1-\alpha}.$$

For $i \in \{Y, XY\}$, define the micro-dilated local increment profile

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} := \varepsilon_{h,\alpha}^{-H_i}\mathfrak{d}_{\varepsilon_{h,\alpha}}\left(h^{-\alpha H_i}\mathcal{D}^{(\alpha)}_{t,h}\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}\right) \in L^2(\mathbb{R}_+). \tag{174}$$

Also define the micro-dilated local transfer operator by

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} : \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}) \subset L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \qquad \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi := \varepsilon^{\beta}_{h,\alpha}\mathfrak{d}_{\varepsilon_{h,\alpha}}\Big(\mathfrak{T}_{t,h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi\Big), \tag{175}$$

with domain

$$\mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}) := \left\{\Phi \in L^2(\mathbb{R}_+) : \mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}\right\}.$$

Proposition 14.9 (Exact micro-local identities)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, and let $\alpha \in (0,1)$. Then for every sufficiently small $h > 0$:*

(i) for $i \in \{Y, XY\}$,

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} = \widehat{g}_{i,t,h}\,\mathbf{1}_{[0,\varepsilon^{-1}_{h,\alpha}]} \qquad \text{in } L^2(\mathbb{R}_+), \tag{176}$$

where $\widehat{g}_{i,t,h}$ is defined in (71)*;*

(ii) for every $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$,

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi = h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi; \tag{177}$$

(iii) for every $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$,

$$h^{-H_{XY}}\mathcal{F}_{t,h}(\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi) = \left\langle \mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}, \Phi\right\rangle_{L^2(\mathbb{R}_+)} - \left\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha}, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\right\rangle_{L^2(\mathbb{R}_+)}. \tag{178}$$

Proof. For $i \in \{Y, XY\}$, $\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}$ is supported in $[t-h^\alpha, t]$, so Proposition 14.3(i) gives

$$\mathfrak{d}_{\varepsilon_{h,\alpha}}\Big(\mathcal{D}^{(\alpha)}_{t,h}\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}\Big) = \mathcal{D}_{t,h}\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}.$$

Multiplying by $\varepsilon^{-H_i}_{h,\alpha}h^{-\alpha H_i} = h^{-H_i}$ yields

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} = h^{-H_i}\mathcal{D}_{t,h}\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}.$$

Because

$$\mathrm{supp}(\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h}) \subset [t-h^\alpha, t],$$

its h-dilation is supported in $[0, \varepsilon^{-1}_{h,\alpha}]$. Therefore

$$h^{-H_i}\mathcal{D}_{t,h}\Pi^{\mathrm{near},\alpha}_{t,h}g_{i,t,h} = \widehat{g}_{i,t,h}\,\mathbf{1}_{[0,\varepsilon^{-1}_{h,\alpha}]},$$

which proves (176).

Part (ii) is exactly Proposition 14.3(iii).

For part (iii), set

$$v_h := \mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi.$$

By (166), one also has

$$v_h = \mathcal{L}^{(\alpha)}_{t,h} \mathfrak{l}_{\varepsilon_{h,\alpha}} \Phi.$$

Since $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$, the right-hand side belongs to $\mathfrak{G}_t$, so $\mathcal{F}_{t,h}(v_h)$ is well-defined. By Definition 7.1, Lemma 9.2, Lemma 7.4, and part (ii),

$$\begin{aligned} h^{-H_{XY}} \mathcal{F}_{t,h}(v_h) &= h^{-H_{XY}} \langle g_{XY,t,h}, v_h \rangle_{H_t} - h^{-H_{XY}} \langle g_{Y,t,h}, \mathcal{S}_t v_h \rangle_{H_t} \\ &= \langle \widehat{g}_{XY,t,h}, \mathfrak{p}_{\varepsilon_{h,\alpha}} \Phi \rangle_{L^2(\mathbb{R}_+)} - \langle \widehat{g}_{Y,t,h}, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \Phi \rangle_{L^2(\mathbb{R}_+)}. \end{aligned}$$

The first term is (88) applied to $\mathfrak{p}_{\varepsilon_{h,\alpha}} \Phi$. For the second term,

$$h^{-H_{XY}} \langle g_{Y,t,h}, \mathcal{S}_t v_h \rangle_{H_t} = \left\langle h^{-H_Y} \mathcal{D}_{t,h} g_{Y,t,h}, h^{\beta} \mathcal{D}_{t,h} \mathcal{S}_t v_h \right\rangle_{L^2(\mathbb{R}_+)},$$

and then applies (177). Because $\widehat{g}_{XY,t,h}$ and $\mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}$ agree on $[0, \varepsilon^{-1}_{h,\alpha}]$ by (176),

$$\langle \widehat{g}_{XY,t,h}, \mathfrak{p}_{\varepsilon_{h,\alpha}} \Phi \rangle_{L^2(\mathbb{R}_+)} = \left\langle \mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}, \Phi \right\rangle_{L^2(\mathbb{R}_+)},$$

and the same argument with $i = Y$ gives

$$\langle \widehat{g}_{Y,t,h}, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \Phi \rangle_{L^2(\mathbb{R}_+)} = \left\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha}, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \Phi \right\rangle_{L^2(\mathbb{R}_+)}.$$

This proves (178). □

Theorem 14.10 (Micro-profile limit inside every h^α layer)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, let $\alpha \in (0,1)$, and let $i \in \{Y, XY\}$. Then*

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} \longrightarrow L_i(t,t) \psi_{H_i} \qquad \text{in } L^2(\mathbb{R}_+) \tag{179}$$

as $h \downarrow 0$.

Proof. By Proposition 14.9(i),

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} = \widehat{g}_{i,t,h} \mathbf{1}_{[0, \varepsilon^{-1}_{h,\alpha}]}.$$

Hence

$$\begin{aligned} \left\| \mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} - L_i(t,t) \psi_{H_i} \right\|_{L^2(\mathbb{R}_+)} &\le \| \widehat{g}_{i,t,h} - L_i(t,t) \psi_{H_i} \|_{L^2(\mathbb{R}_+)} \\ &\quad + |L_i(t,t)| \left\| \psi_{H_i} \mathbf{1}_{(\varepsilon^{-1}_{h,\alpha}, \infty)} \right\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

The first term tends to 0 by Proposition 7.5. Since $\varepsilon^{-1}_{h,\alpha} = h^{\alpha - 1} \to \infty$ and $\psi_{H_i} \in L^2(\mathbb{R}_+)$, the second term also tends to 0. This proves (179). □

Remark 14.11 (Interpretation of the two-scale analysis). The exact local transfer family on the unit interval is an intermediate asymptotic object for the gain functional. After the first localization to a boundary layer of width h^α, the singular coefficients still collapse onto the smaller internal scale $\varepsilon_{h,\alpha} = h^{1-\alpha}$ near $u = 0$. Proposition 14.9 identifies the micro-dilated family $\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}$, rather than $\mathfrak{T}_{t,h,\alpha}$ alone, as the operator governing the singular gain. Corollary 14.6 identifies that family with the micro-dilated Volterra pair $\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}, \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}$. The argument then compares $\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}$ with the half-line fractional model $\mathcal{T}^{\mathrm{mod}}_{t,\beta}$ on graph-bounded families and lifts that micro-local control to the global supremum in Proposition 7.2.

15. Strong micro-local convergence on the smooth core

The second blow-up produces the half-line tangent problem. That tangent problem controls the exact micro-transfer family on fixed smooth compactly supported profiles. This is the half-line analogue of the unit-interval smooth-core theorem from Section 12.

Definition 15.1 (Half-line right fractional integral on compact support)**.** For $\lambda \in (0,1)$ and $f \in L^2(\mathbb{R}_+)$ with compact support, define

$$(\mathfrak{I}^{\lambda}_{-} f)(r) := \frac{1}{\Gamma(\lambda)} \int_r^\infty (q - r)^{\lambda - 1} f(q)\, dq, \qquad r \ge 0. \tag{180}$$

Lemma 15.2 (Half-line semigroup and injectivity on compact supports). *Let $\lambda, \mu \in (0,1)$ with $\lambda + \mu \le 1$, and let $f \in L^2(\mathbb{R}_+)$ have compact support. Then:*

(i)

$$\mathfrak{I}_-^{\lambda}\mathfrak{I}_-^{\mu} f = \mathfrak{I}_-^{\lambda+\mu} f; \tag{181}$$

(ii) if $\mathfrak{I}_-^{\lambda} f = 0$, then $f = 0$ almost everywhere.

Proof. For almost every $r \ge 0$, Fubini's theorem gives

$$\begin{aligned}(\mathfrak{I}_-^{\lambda}\mathfrak{I}_-^{\mu} f)(r) &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_r^\infty \int_s^\infty (s-r)^{\lambda-1}(q-s)^{\mu-1} f(q)\,dq\,ds \\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_r^\infty \left(\int_r^q (s-r)^{\lambda-1}(q-s)^{\mu-1}\,ds\right) f(q)\,dq.\end{aligned}$$

With $s = r + (q-r)\theta$, the inner integral is

$$(q-r)^{\lambda+\mu-1}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)},$$

which proves (181).

For injectivity, assume $\mathfrak{I}_-^{\lambda} f = 0$. Applying (181) with $\mu = 1-\lambda$ yields

$$\mathfrak{I}_-^{1} f = 0.$$

Since

$$(\mathfrak{I}_-^{1} f)(r) = \int_r^\infty f(q)\,dq,$$

the function $\mathfrak{I}_-^{1} f$ is absolutely continuous with derivative $-f$. Hence $f = 0$ almost everywhere. □

Proposition 15.3 (Conjugated target-side micro operator on fixed compact supports). *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and $R > 0$, and set*

$$\lambda := H_Y + \frac{1}{2} \in (0,1), \qquad E_R := L^2([0,R]),$$

identified with its zero extension to $L^2(\mathbb{R}_+)$. Then for all sufficiently small $h > 0$ satisfying

$$hR < t, \qquad R < \varepsilon_{h,\alpha}^{-1},$$

there exists a bounded operator

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} \in \mathcal{L}(E_R)$$

such that:

(i) for every $\Psi \in E_R$, the function $\mathfrak{I}_-^{1-\lambda}(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi)$ is absolutely continuous on $[0,\infty)$, and

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^{\lambda}(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi) = \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi \qquad \text{in } E_R; \tag{182}$$

(ii) there exists $C_{t,R} < \infty$ such that

$$\left\|\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} - L_Y(t,t) I_{E_R}\right\|_{\mathcal{L}(E_R)} \le C_{t,R} h \tag{183}$$

for all sufficiently small $h > 0$.

Proof. For $0 \le r \le q \le R$, define

$$a_{Y,t,h}(r,q) := L_Y(t-hr, t-hq).$$

Let $\Psi \in E_R$. Because Ψ is supported in $[0,R]$ and $R < \varepsilon_{h,\alpha}^{-1}$, (168) gives

$$(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi)(r) = \int_r^R (q-r)^{\lambda-1} a_{Y,t,h}(r,q)\Psi(q)\,dq, \qquad r \ge 0,$$

with the understanding that the right-hand side is 0 for $r > R$.

Set

$$f_h := \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi.$$

For $0 \le r \le R$, Definition 15.1 and Fubini's theorem give

$$(\mathfrak{I}_-^{1-\lambda} f_h)(r) = \frac{1}{\Gamma(1-\lambda)} \int_r^R \int_s^R (s-r)^{-\lambda} (q-s)^{\lambda-1} a_{Y,t,h}(s,q) \Psi(q)\, dq\, ds$$
$$= \int_r^R F^{\mathrm{mic}}_{Y,t,h,R}(r,q) \Psi(q)\, dq,$$

where

$$F^{\mathrm{mic}}_{Y,t,h,R}(r,q) := \frac{1}{\Gamma(1-\lambda)} \int_0^1 s^{-\lambda} (1-s)^{\lambda-1} a_{Y,t,h}(r+(q-r)s, q)\, ds.$$

Since L_Y is C^1, the function $F^{\mathrm{mic}}_{Y,t,h,R}$ is continuous on $\Delta_R := \{0 \le r \le q \le R\}$, and its first derivative in r is bounded. Therefore $\mathfrak{I}_-^{1-\lambda} f_h$ is absolutely continuous on $[0,R]$, vanishes on $[R,\infty)$, and differentiation under the integral sign yields for almost every $r \in [0,R]$,

$$\mathfrak{D}_-^{\lambda} f_h(r) = \Gamma(\lambda) a_{Y,t,h}(r,r) \Psi(r) - \int_r^R G^{\mathrm{mic}}_{Y,t,h,R}(r,q) \Psi(q)\, dq,$$

where

$$G^{\mathrm{mic}}_{Y,t,h,R}(r,q) := \frac{\Gamma(\lambda)}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 s^{-\lambda} (1-s)^{\lambda} \partial_1 a_{Y,t,h}(r+(q-r)s, q)\, ds.$$

Define

$$(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} \Psi)(r) := a_{Y,t,h}(r,r)\Psi(r)$$
$$- \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_r^R \int_0^1 s^{-\lambda}(1-s)^{\lambda} \partial_1 a_{Y,t,h}(r+(q-r)s, q)\, ds\, \Psi(q)\, dq$$

for $0 \le r \le R$. This proves (182).

To prove (183), note first that

$$|a_{Y,t,h}(r,r) - L_Y(t,t)| \le C_{t,R} h \qquad \text{for } 0 \le r \le R,$$

because L_Y is C^1. Also,

$$\partial_1 a_{Y,t,h}(x,q) = -h\, \partial_1 L_Y(t-hx, t-hq),$$

so $|\partial_1 a_{Y,t,h}(x,q)| \le C_{t,R} h$ uniformly on Δ_R. Set

$$D_{t,h,R} := \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} - a_{Y,t,h}(\cdot,\cdot)_{\mathrm{diag}} I_{E_R}.$$

Then $D_{t,h,R}$ has integral kernel bounded by $C_{t,R} h \mathbf{1}_{\Delta_R}$. Its Hilbert–Schmidt norm is $O_{t,R}(h)$, which proves (183). □

Corollary 15.4 (Uniform invertibility of the conjugated micro target operator)**.** *Under the assumptions of Proposition 15.3, there exist $h_0 \in (0,t]$ and $C_{t,R} < \infty$ such that for every $0 < h \le h_0$,*

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} : E_R \to E_R$$

is invertible and

$$\left\| (\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R})^{-1} \right\|_{\mathcal{L}(E_R)} \le C_{t,R}. \tag{184}$$

Proof. By Assumption 3.1, $L_Y(t,t) > 0$. Proposition 15.3 gives

$$\left\| \frac{1}{L_Y(t,t)} \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R} - I_{E_R} \right\|_{\mathcal{L}(E_R)} \le \frac{C_{t,R}}{L_Y(t,t)} h.$$

For h small enough the right-hand side is $< 1/2$, so the Neumann series gives invertibility and (184). □

Proposition 15.5 (Conjugated micro source asymptotics on the half-line smooth core)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Let $\Phi \in C_c^\infty((0,\infty))$*, and choose* $R > 0$ *such that*

$$\operatorname{supp}\Phi \subset [0,R).$$

Then there exists $C_{\Phi,t,R} < \infty$ *such that for all sufficiently small* $h > 0$ *with* $hR < t$ *and* $R < \varepsilon_{h,\alpha}^{-1}$*,*

$$\left\| \frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^{\lambda} (\mathfrak{V}_{XY,t,h,\alpha}^{\mathrm{mic}} \Phi) - L_Y(t,t) \mathcal{T}_{t,\beta}^{\mathrm{mod}} \Phi \right\|_{L^2(\mathbb{R}_+)} \le C_{\Phi,t,R} h. \tag{185}$$

Proof. Define

$$a_{XY,t,h}(r,q) := L_{XY}(t-hr, t-hq), \qquad 0 \le r \le q \le R.$$

For $0 \le r \le R$,

$$(\mathfrak{V}_{XY,t,h,\alpha}^{\mathrm{mic}} \Phi)(r) = \int_r^R (q-r)^{\mu-1} a_{XY,t,h}(r,q) \Phi(q)\, dq,$$

and the left-hand side vanishes for $r > R$.

Using Definition 15.1 and the change of variables $s = r + (q-r)\theta$, one obtains for almost every $r \in [0,R]$,

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^{\lambda} (\mathfrak{V}_{XY,t,h,\alpha}^{\mathrm{mic}} \Phi)(r) = -\frac{d}{dr} \int_r^R (q-r)^{-\beta} g_{t,h,R}^{\mathrm{mic}}(r,q) \Phi(q)\, dq, \tag{186}$$

where

$$g_{t,h,R}^{\mathrm{mic}}(r,q) := \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 \theta^{-\lambda} (1-\theta)^{\mu-1} a_{XY,t,h}(r + (q-r)\theta, q)\, d\theta.$$

Because L_{XY} is C^1, one has

$$\|g_{t,h,R}^{\mathrm{mic}} - g_t^{\mathrm{hl}}\|_{C^1(\Delta_R)} \le C_{t,R} h, \tag{187}$$

where

$$g_t^{\mathrm{hl}} := \frac{L_{XY}(t,t)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(1-\beta)} = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}.$$

Now write

$$Q_{t,h,\Phi}^{\mathrm{mic}}(r) := \frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^{\lambda} (\mathfrak{V}_{XY,t,h,\alpha}^{\mathrm{mic}} \Phi)(r) - L_Y(t,t) \mathcal{T}_{t,\beta}^{\mathrm{mod}} \Phi(r).$$

Using (186), the constancy of g_t^{hl}, and (83), this becomes

$$Q_{t,h,\Phi}^{\mathrm{mic}}(r) = -\frac{d}{dr} \int_r^R (q-r)^{-\beta} (g_{t,h,R}^{\mathrm{mic}}(r,q) - g_t^{\mathrm{hl}}) \Phi(q)\, dq.$$

Since $\operatorname{supp}\Phi \subset [0,R)$, there is no boundary term at $q = R$. With $q = r + u$,

$$Q_{t,h,\Phi}^{\mathrm{mic}}(r) = -\int_0^{R-r} u^{-\beta} \Big[(\partial_1 + \partial_2)(g_{t,h,R}^{\mathrm{mic}} - g_t^{\mathrm{hl}})(r, r+u)\, \Phi(r+u) + (g_{t,h,R}^{\mathrm{mic}} - g_t^{\mathrm{hl}})(r, r+u)\, \Phi'(r+u) \Big]\, du.$$

By (187), the bracketed term is $O_{\Phi,t,R}(h)$ uniformly in r and u. Since $\beta \in (0,1)$,

$$\int_0^{R-r} u^{-\beta}\, du \le \frac{R^{1-\beta}}{1-\beta} \qquad \text{for } 0 \le r \le R.$$

Hence $\|Q_{t,h,\Phi}^{\mathrm{mic}}\|_{L^\infty([0,R])} \le C_{\Phi,t,R} h$, which proves (185). □

Theorem 15.6 (Strong micro-local convergence on the half-line smooth core)**.** *Assume Assumptions 3.1 and 4.1, let* $\eta > 0$*, fix* $t \in (0,T)$*,* $\alpha \in (0,1)$*, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then for every $\Phi \in C_c^\infty((0,\infty))$ *there exist* $R_\Phi < \infty$*,* $h_\Phi \in (0,t]$*, and* $C_\Phi < \infty$ *such that for every* $0 < h \le h_\Phi$*,*

$$\operatorname{supp}\Phi \subset [0, R_\Phi), \qquad \Phi \in \mathcal{D}(\mathfrak{T}_{t,h,\alpha}^{\mathrm{mic}}),$$

$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi$ *is supported in* $[0,R_\Phi]$, *and*

$$\left\|\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi - \mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi\right\|_{L^2(\mathbb{R}_+)} \le C_\Phi h. \tag{188}$$

Proof. Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Fix $\Phi \in C_c^\infty((0,\infty))$, and choose $R_\Phi < \infty$ such that

$$\operatorname{supp}\Phi \subset [0,R_\Phi).$$

For $h>0$ sufficiently small, define

$$q^{\mathrm{mic}}_{t,h,\Phi} := \frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_-\left(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi\right) \in E_{R_\Phi}.$$

By Corollary 15.4, the operator $\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_\Phi}$ is invertible for small h, so we may set

$$\Psi_h := \left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_\Phi}\right)^{-1} q^{\mathrm{mic}}_{t,h,\Phi} \in E_{R_\Phi},$$

again identified with its zero extension to $L^2(\mathbb{R}_+)$.

Using Proposition 15.5, Proposition 15.3, and (184),

$$\begin{aligned}\left\|\Psi_h - \mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi\right\|_{E_{R_\Phi}} &\le \left\|\left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_\Phi}\right)^{-1}\right\|_{\mathcal{L}(E_{R_\Phi})}\Big(\left\|q^{\mathrm{mic}}_{t,h,\Phi} - L_Y(t,t)\mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi\right\|_{E_{R_\Phi}} \\ &\qquad + \left\|\left(L_Y(t,t)I_{E_{R_\Phi}} - \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_\Phi}\right)\mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi\right\|_{E_{R_\Phi}}\Big) \\ &\le C_\Phi h.\end{aligned}$$

Since $\mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi$ is supported in $[0,R_\Phi]$, this already gives

$$\left\|\Psi_h - \mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi\right\|_{L^2(\mathbb{R}_+)} \le C_\Phi h. \tag{189}$$

Now set

$$r_h := \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_h - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi.$$

Because both Ψ_h and Φ are supported in $[0,R_\Phi]$, so is r_h. By the definitions of Ψ_h and $q^{\mathrm{mic}}_{t,h,\Phi}$, together with Proposition 15.3,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_- r_h = \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_\Phi}\Psi_h - q^{\mathrm{mic}}_{t,h,\Phi} = 0 \qquad \text{in } E_{R_\Phi}.$$

Hence, by Proposition 15.3 for the first term and the proof of Proposition 15.5 for the second, $\mathfrak{I}^{1-\lambda}_- r_h$ is absolutely continuous with zero derivative, so it is constant on $[0,\infty)$. Since r_h is compactly supported, one has

$$(\mathfrak{I}^{1-\lambda}_- r_h)(r) = 0 \qquad \text{for } r \ge R_\Phi,$$

and therefore $\mathfrak{I}^{1-\lambda}_- r_h \equiv 0$. By Lemma 15.2, this implies

$$r_h = 0,$$

that is,

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_h = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi. \tag{190}$$

Set $\varepsilon := \varepsilon_{h,\alpha}$. Since $R_\Phi < \varepsilon^{-1}$ for h small, Proposition 14.5 and (190) give

$$\mathfrak{d}_\varepsilon \mathfrak{A}_{Y,t,h,\alpha}\mathfrak{l}_\varepsilon \Psi_h = \varepsilon^{H_Y}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_h = \varepsilon^\beta \mathfrak{d}_\varepsilon \mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon\Phi.$$

Applying $\mathfrak{l}_\varepsilon$ and using $\mathfrak{l}_\varepsilon\mathfrak{d}_\varepsilon = I_E$ from Lemma 14.2 yields

$$\mathfrak{A}_{Y,t,h,\alpha}\left(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon\Psi_h\right) = \mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon\Phi. \tag{191}$$

Now define

$$w_h := \mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_\varepsilon\Phi \in H^{\mathrm{near}}_{t,h,\alpha}, \qquad u_h := h^{-\alpha\beta}\mathcal{L}^{(\alpha)}_{t,h}\left(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon\Psi_h\right) \in H^{\mathrm{near}}_{t,h,\alpha}.$$

Using Proposition 11.6 and (191),

$$\begin{aligned}\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}u_h &= h^{-\alpha\beta+\alpha H_Y}\mathfrak{A}_{Y,t,h,\alpha}(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon\Psi_h)\\ &= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon\Phi\\ &= \mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{XY,t}w_h.\end{aligned}$$

Both $\mathcal{K}_{Y,t}u_h$ and $\mathcal{K}_{XY,t}w_h$ are supported in $[t-h^\alpha, t]$, so Lemma 11.2 implies

$$\mathcal{K}_{Y,t}u_h = \mathcal{K}_{XY,t}w_h.$$

Hence $w_h \in \mathcal{D}(\mathcal{S}_t)$, $\mathcal{S}_t w_h = u_h$, and

$$\mathfrak{l}_\varepsilon\Phi = \mathcal{D}^{(\alpha)}_{t,h}w_h \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}.$$

Therefore $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$, and by Definitions 11.7 and 14.8,

$$\begin{aligned}\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi &= \varepsilon^\beta\mathfrak{d}_\varepsilon\mathfrak{T}_{t,h,\alpha}\mathfrak{l}_\varepsilon\Phi\\ &= \varepsilon^\beta\mathfrak{d}_\varepsilon(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon\Psi_h)\\ &= \mathfrak{d}_\varepsilon\mathfrak{l}_\varepsilon\Psi_h = \mathfrak{p}_\varepsilon\Psi_h = \Psi_h,\end{aligned}$$

because $\operatorname{supp}\Psi_h \subset [0, R_\Phi] \subset [0, \varepsilon^{-1}]$. Combining this identity with (189) proves (188). □

16. Fixed-window micro graph theory

The half-line micro family remains globally nonlocal because the right-sided transfer law depends on the full future tail. On each fixed window $[0, R]$, the exact micro family closes: inputs supported in $[0, R]$ produce outputs supported in $[0, R]$. The problem becomes a finite-interval graph convergence problem. Tail leakage is the only nonlocal obstruction to a global theorem.

Corollary 16.1 (General support preservation near the terminal time)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0, T)$, and let $a \in (0, t)$. If $w \in \mathfrak{G}_t$ satisfies*

$$\operatorname{supp}(w) \subset [a, t],$$

then

$$\operatorname{supp}(\mathcal{S}_t w) \subset [a, t].$$

Proof. The argument is identical to that of Proposition 11.3, applied to the truncated operator $\mathcal{K}_{Y,a}$. □

Definition 16.2 (Fixed-window half-line Volterra model and graph norm)**.** Assume

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

fix $t \in (0, T)$, and let $R > 0$. Set

$$E_R := L^2([0, R]),$$

identified with its zero extension to $L^2(\mathbb{R}_+)$. For $H \in (0, \frac{1}{2})$, define

$$\mathfrak{J}_{H,R} : E_R \to E_R, \qquad (\mathfrak{J}_{H,R}\Phi)(r) := \int_r^R (q-r)^{H-\frac{1}{2}}\Phi(q)\,dq, \qquad 0 \le r \le R.$$

Define the fixed-window model transfer domain

$$\mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}) := \{\Phi \in E_R : \mathfrak{J}_{H_{XY},R}\Phi \in \operatorname{Ran}(\mathfrak{J}_{H_Y,R})\},$$

and the corresponding transfer operator

$$\mathfrak{T}^{\mathrm{hl}}_{t,R} : \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}) \subset E_R \to E_R, \qquad \mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi := \frac{L_{XY}(t,t)}{L_Y(t,t)}\mathfrak{J}^{-1}_{H_Y,R}\mathfrak{J}_{H_{XY},R}\Phi.$$

For $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$, define the fixed-window model graph norm

$$\|\Phi\|^2_{\mathfrak{G}^{\mathrm{hl}}_{t,R}} := \|\Phi\|^2_{E_R} + \frac{\eta^2}{\nu_Y^2}\|\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi\|^2_{E_R}. \tag{192}$$

Theorem 16.3 (Fractional-derivative representation of the fixed-window half-line model)**.** *Assume the hypotheses of Definition 16.2. Then:*

(i) for every $\Phi \in C_c^\infty((0,R))$,

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = \mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi \qquad \text{in } E_R; \tag{193}$$

(ii) $\mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}) = H^\beta((0,R))$ as sets;

(iii) the graph norm $\|\cdot\|_{\mathfrak{G}^{\mathrm{hl}}_{t,R}}$ is equivalent to the $H^\beta((0,R))$ norm;

(iv) $C_c^\infty((0,R))$ is dense in $\mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$ with respect to $\|\cdot\|_{\mathfrak{G}^{\mathrm{hl}}_{t,R}}$.

Proof. Define the unitary scaling map

$$U_R : E_R \to E = L^2([0,1]), \qquad (U_R\Phi)(u) := R^{1/2}\Phi(Ru), \qquad 0 \le u \le 1.$$

For $H \in (0,\frac{1}{2})$ and $\Phi \in E_R$, a change of variables gives

$$U_R \mathfrak{J}_{H,R} U_R^{-1} = R^{H+\frac{1}{2}}\mathfrak{J}_H. \tag{194}$$

In particular, $\mathfrak{J}_{H,R}$ is injective because $\mathfrak{J}_H$ is injective by Lemma 11.10. Hence $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$ if and only if $U_R\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{fr}}_t)$, and in that case

$$U_R \mathfrak{T}^{\mathrm{hl}}_{t,R} U_R^{-1} = R^{-\beta}\mathfrak{T}^{\mathrm{fr}}_t. \tag{195}$$

Now let $\Phi \in C_c^\infty((0,R))$. Then $U_R\Phi \in C_c^\infty((0,1))$, so Proposition 11.12(ii) gives

$$\mathfrak{T}^{\mathrm{fr}}_t(U_R\Phi) = \mathbf{c}_t\,\mathfrak{D}^\beta_{-,1}(U_R\Phi).$$

Because Φ vanishes near R, a direct change of variables in the definition of the right-sided derivative yields

$$\mathfrak{D}^\beta_{-,1}(U_R\Phi) = R^{\beta+\frac{1}{2}}U_R(\mathfrak{D}^\beta_{-}\Phi).$$

Combining this with (195) gives

$$U_R\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = R^{-\beta}\mathbf{c}_t\mathfrak{D}^\beta_{-,1}(U_R\Phi) = \mathbf{c}_t U_R(\mathfrak{D}^\beta_{-}\Phi) = U_R(\mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi),$$

which proves (193).

Items (ii)–(iv) follow from (195), Theorem 12.15, and the fact that U_R is a topological isomorphism between $H^\beta((0,R))$ and $H^\beta((0,1))$. □

Corollary 16.4 (Fixed-window model functional vanishes on the full graph domain)**.** *Assume the hypotheses of Theorem 16.3. Define*

$$\Lambda^{\mathrm{hl}}_{t,R}(\Phi) := L_{XY}(t,t)\langle\psi_{H_{XY}},\Phi\rangle_{L^2(\mathbb{R}_+)} - L_Y(t,t)\left\langle\psi_{H_Y},\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi\right\rangle_{L^2(\mathbb{R}_+)}, \qquad \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}).$$

Then $\Lambda^{\mathrm{hl}}_{t,R}$ is continuous on $\mathfrak{G}^{\mathrm{hl}}_{t,R}$, and

$$\Lambda^{\mathrm{hl}}_{t,R}(\Phi) = 0 \qquad \text{for every } \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}).$$

Proof. Continuity follows from Cauchy–Schwarz, $\psi_{H_i} \in L^2(\mathbb{R}_+)$, and (192). By Theorem 16.3(iv), choose $\Phi_n \in C_c^\infty((0,R))$ such that

$$\|\Phi_n - \Phi\|_{\mathfrak{G}^{\mathrm{hl}}_{t,R}} \to 0.$$

By Theorem 16.3(i), one has

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi_n = \mathcal{T}^{\mathrm{mod}}_{t,\beta}\Phi_n,$$

so Corollary 8.5 gives $\Lambda^{\mathrm{hl}}_{t,R}(\Phi_n) = 0$ for every n. Passing to the limit by continuity proves the claim. □

Definition 16.5 (Exact micro graph space on a fixed window)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0, T)$, $\alpha \in (0, 1)$, $R > 0$, and $h > 0$ such that

$$hR < t, \qquad R < \varepsilon_{h,\alpha}^{-1}.$$

Define the exact micro graph space

$$\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R} := \left\{ \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}) : \operatorname{supp} \Phi \subset [0, R] \right\}.$$

For $\Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}$, define

$$\|\Phi\|^2_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}} := \|\Phi\|^2_{E_R} + \frac{\eta^2}{\nu_Y^2} \|\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\|^2_{E_R}. \tag{196}$$

For $i \in \{Y, XY\}$, define the fixed-window exact micro Volterra operator

$$\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha,R} : E_R \to E_R$$

by

$$(\mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha,R}\Phi)(r) := \int_r^R (q - r)^{H_i - \frac{1}{2}} L_i(t - hr, t - hq)\Phi(q)\, dq, \qquad 0 \le r \le R.$$

Proposition 16.6 (Exact micro transfer closes on every fixed window)**.** *Assume the hypotheses of Definition 16.5. Then for every $\Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}$:*

(i)

$$\operatorname{supp}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) \subset [0, R]; \tag{197}$$

(ii)

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi \qquad \textit{in } E_R. \tag{198}$$

Proof. Set $\varepsilon := \varepsilon_{h,\alpha}$, and define

$$w_h := \mathcal{L}_{t,h}\Phi.$$

Because $\operatorname{supp} \Phi \subset [0, R]$ and $R < \varepsilon^{-1}$, one has

$$\mathfrak{p}_\varepsilon \Phi = \Phi$$

and therefore, by (166),

$$w_h = \mathcal{L}_{t,h}\Phi = \mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_\varepsilon\Phi.$$

In particular, $w_h \in \mathfrak{G}_t$, and

$$\operatorname{supp}(w_h) \subset [t - hR, t].$$

Corollary 16.1 with $a = t - hR$ therefore gives

$$\operatorname{supp}(\mathcal{S}_t w_h) \subset [t - hR, t].$$

By Proposition 14.9(ii),

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi = h^\beta \mathcal{D}_{t,h}\mathcal{S}_t w_h.$$

Since $\mathcal{S}_t w_h$ is supported in $[t - hR, t]$, its h-dilation is supported in $[0, R]$, proving (197).

Corollary 14.6 gives

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi) = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi \qquad \text{in } L^2(\mathbb{R}_+).$$

Because both Φ and $\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi$ are supported in $[0, R]$, the operators on both sides coincide with their fixed-window restrictions on E_R, which proves (198). $\square$

Proposition 16.7 (Operator-norm freeze of the exact micro Volterra pair on fixed windows)**.** *Assume Assumption 3.1, fix $t \in (0, T)$, $\alpha \in (0, 1)$, and $R > 0$, and let $i \in \{Y, XY\}$. Then there exists $C_{t,R} < \infty$ such that for all sufficiently small $h > 0$ with $hR < t$,*

$$\left\| \mathfrak{V}^{\mathrm{mic}}_{i,t,h,\alpha,R} - L_i(t, t)\mathfrak{J}_{H_i,R} \right\|_{\mathcal{L}(E_R)} \le C_{t,R}h. \tag{199}$$

Proof. The difference kernel is

$$K^{\mathrm{err}}_{i,t,h,R}(r,q) := (q-r)^{H_i-\frac{1}{2}}\big(L_i(t-hr,t-hq)-L_i(t,t)\big)\mathbf{1}_{\{0\le r\le q\le R\}}.$$

Because L_i is C^1 near (t,t), one has

$$|L_i(t-hr,t-hq)-L_i(t,t)|\le C_{t,R}h \qquad \text{for } 0\le r\le q\le R$$

when $hR<t$. Hence

$$|K^{\mathrm{err}}_{i,t,h,R}(r,q)|\le C_{t,R}h(q-r)^{H_i-\frac{1}{2}}\mathbf{1}_{\{0\le r\le q\le R\}}.$$

Since this majorant is Hilbert–Schmidt on Δ_R, the Hilbert–Schmidt norm of the difference is $O_{t,R}(h)$, which proves (199). □

Proposition 16.8 (Weak graph compactness of the exact micro family on fixed windows)**.** *Assume Assumptions 3.1 and 4.1, let $\eta>0$, fix $t\in(0,T)$, $\alpha\in(0,1)$, $R>0$, and suppose*

$$0<H_{XY}<H_Y<\tfrac{1}{2}.$$

Let $h_n\downarrow 0$ with $h_nR<t$ and $R<\varepsilon^{-1}_{h_n,\alpha}$ for every n, and let

$$\Phi_n\in\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha,R}$$

satisfy

$$\sup_{n\ge1}\|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha,R}}<\infty.$$

Then there exist a subsequence, still denoted (Φ_n), and some

$$\Phi\in\mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$$

such that

$$\Phi_n\rightharpoonup\Phi \qquad \textit{weakly in } E_R,$$
$$\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n\rightharpoonup\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi \qquad \textit{weakly in } E_R.$$

Moreover,

$$\|\Phi\|_{\mathfrak{G}^{\mathrm{hl}}_{t,R}}\le\liminf_{n\to\infty}\|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha,R}}.$$

Proof. By Definition 16.5 and Proposition 16.6(i), both (Φ_n) and $(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n)$ are bounded in E_R. Passing to a subsequence, assume

$$\Phi_n\rightharpoonup\Phi, \qquad \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n\rightharpoonup\Psi \qquad \text{weakly in } E_R$$

for some $\Phi,\Psi\in E_R$.

By Proposition 16.6(ii),

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h_n,\alpha,R}\big(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n\big)=\mathfrak{V}^{\mathrm{mic}}_{XY,t,h_n,\alpha,R}\Phi_n.$$

Using Proposition 16.7 exactly as in the proof of Proposition 11.15, one obtains

$$\mathfrak{V}^{\mathrm{mic}}_{XY,t,h_n,\alpha,R}\Phi_n\rightharpoonup L_{XY}(t,t)\mathfrak{J}_{H_{XY},R}\Phi \qquad \text{weakly in } E_R$$

and

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h_n,\alpha,R}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n)\rightharpoonup L_Y(t,t)\mathfrak{J}_{H_Y,R}\Psi \qquad \text{weakly in } E_R.$$

Passing to the weak limit in the exact identity gives

$$L_Y(t,t)\mathfrak{J}_{H_Y,R}\Psi=L_{XY}(t,t)\mathfrak{J}_{H_{XY},R}\Phi.$$

By Definition 16.2, this means $\Phi\in\mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$, and by injectivity of $\mathfrak{J}_{H_Y,R}$,

$$\Psi=\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi.$$

Weak lower semicontinuity of the norm in E_R then yields the graph-norm liminf exactly as in Proposition 11.15. □

Corollary 16.9 (Fixed-window micro Mosco convergence)**.** *Assume the hypotheses of Proposition 16.8. Let $h_n \downarrow 0$ with $h_n R < t$ and $R < \varepsilon_{h_n,\alpha}^{-1}$ for every n. Define*

$$\Gamma^{\mathrm{mic}}_{t,h_n,\alpha,R} := \left\{ (\Phi, \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi) : \Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha,R} \right\} \subset E_R \times E_R,$$

and define the fixed-window model graph

$$\Gamma^{\mathrm{hl}}_{t,R} := \left\{ (\Phi, \mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi) : \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R}) \right\}.$$

Then $\Gamma^{\mathrm{mic}}_{t,h_n,\alpha,R} \to \Gamma^{\mathrm{hl}}_{t,R}$ in the sequential Mosco sense.

Proof. The weak-compactness direction is Proposition 16.8. For the recovery direction, Theorem 16.3(iv) gives density of $C_c^\infty((0,R))$ in the fixed-window model graph norm, while Theorem 15.6 gives strong convergence of $\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}$ to $\mathcal{T}^{\mathrm{mod}}_{t,\beta}$ on that smooth core, which coincides with $\mathfrak{T}^{\mathrm{hl}}_{t,R}$ by Theorem 16.3(i). The same diagonal argument as in Theorem 11.18 and Corollary 11.19 therefore yields the Mosco recovery property on $E_R \times E_R$. □

17. Buffered micro tail forcing

Fixed-window theory isolates the remaining nonlocal obstruction. Truncating a global micro pair at an outer radius R breaks the exact Volterra identity on an inner window $[0, R_0]$ only through the tail beyond R. The defect is written exactly. After the same right-sided fractional conjugation used in Section 15, the resulting forcing is quantitatively small.

Definition 17.1 (Buffered micro tail operators)**.** Assume Assumption 3.1, fix $t \in (0, T)$, $\alpha \in (0, 1)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and let $0 < R_0 < R$. For $h > 0$ sufficiently small that $hR < t$ and $R < \varepsilon_{h,\alpha}^{-1}$, define

$$F_{h,R} := L^2([R, \varepsilon_{h,\alpha}^{-1}]),$$

identified with its zero extension to $L^2(\mathbb{R}_+)$. For $\Theta \in F_{h,R}$, define the target-side and source-side buffered micro tail operators on $L^2(\mathbb{R}_+)$ by

$$(\mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\Theta)(r) := \int_{\max\{R,r\}}^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\lambda-1} L_Y(t-hr, t-hq)\Theta(q)\, dq, \tag{200}$$

$$(\mathfrak{W}^{\mathrm{tail}}_{XY,t,h,\alpha,R_0,R}\Theta)(r) := \int_{\max\{R,r\}}^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq)\Theta(q)\, dq \tag{201}$$

for $r \geq 0$. On the inner window $[0, R_0]$, these reduce to (200) and (201) with lower limit R, since $R_0 < R$.

Proposition 17.2 (Conjugated target-side buffered tail forcing)**.** *Assume the hypotheses of Definition 17.1. Then for every sufficiently small $h > 0$ there exists a bounded operator*

$$\mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R} \in \mathcal{L}(F_{h,R}, E_{R_0})$$

such that for every $\Theta \in F_{h,R}$,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\Theta) = \mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\Theta \qquad \textit{in } E_{R_0}. \tag{202}$$

Moreover, there exists $C_{t,R_0} < \infty$ such that

$$\left\| \mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R} \right\|_{\mathcal{L}(F_{h,R}, E_{R_0})} \leq C_{t,R_0} h \sqrt{\varepsilon_{h,\alpha}^{-1} - R} \leq C_{t,R_0} h^{\frac{1+\alpha}{2}} \tag{203}$$

for all sufficiently small $h > 0$.

Proof. For $0 \le r \le R_0$ and $q \in [R, \varepsilon_{h,\alpha}^{-1}]$, define

$$a_{Y,t,h}(r,q) := L_Y(t-hr, t-hq).$$

Since $q - r \ge R - R_0 > 0$, the function

$$k_{Y,t,h,q}(r) := (q-r)^{\lambda-1} a_{Y,t,h}(r,q)$$

is C^1 in r on $[0, R_0]$. The same computation as in the proof of Proposition 15.3, now with fixed upper endpoint q, gives

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^\lambda k_{Y,t,h,q}(r) = K_{Y,t,h}^{\mathrm{tail}}(r,q) \qquad \text{for } 0 \le r \le R_0,$$

where

$$K_{Y,t,h}^{\mathrm{tail}}(r,q) := -\frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 s^{-\lambda}(1-s)^\lambda \partial_1 a_{Y,t,h}(r+(q-r)s, q)\, ds.$$

Hence Fubini's theorem yields

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^\lambda (\mathfrak{W}_{Y,t,h,\alpha,R_0,R}^{\mathrm{tail}} \Theta)(r) = \int_R^{\varepsilon_{h,\alpha}^{-1}} K_{Y,t,h}^{\mathrm{tail}}(r,q)\Theta(q)\, dq$$

for almost every $r \in [0, R_0]$. This defines the bounded operator

$$(\mathfrak{Q}_{Y,t,h,\alpha,R_0,R}^{\mathrm{tail}} \Theta)(r) := \int_R^{\varepsilon_{h,\alpha}^{-1}} K_{Y,t,h}^{\mathrm{tail}}(r,q)\Theta(q)\, dq, \qquad 0 \le r \le R_0,$$

and proves (202).

Because L_Y is C^1,

$$\partial_1 a_{Y,t,h}(x,q) = -h\, \partial_1 L_Y(t-hx, t-hq),$$

so $|\partial_1 a_{Y,t,h}(x,q)| \le C_t h$ uniformly on $\{0 \le x \le q \le \varepsilon_{h,\alpha}^{-1}\}$. Therefore

$$|K_{Y,t,h}^{\mathrm{tail}}(r,q)| \le C_t h \qquad \text{for } 0 \le r \le R_0, \quad R \le q \le \varepsilon_{h,\alpha}^{-1}.$$

The kernel is Hilbert–Schmidt, with

$$\left\| K_{Y,t,h}^{\mathrm{tail}} \right\|_{L^2([0,R_0]\times[R,\varepsilon_{h,\alpha}^{-1}])} \le C_{t,R_0} h \sqrt{\varepsilon_{h,\alpha}^{-1} - R},$$

which proves the first bound in (203). Since $\varepsilon_{h,\alpha}^{-1} = h^{\alpha-1}$, one also has

$$h\sqrt{\varepsilon_{h,\alpha}^{-1} - R} \le h\sqrt{\varepsilon_{h,\alpha}^{-1}} = h^{\frac{1+\alpha}{2}},$$

which proves the second bound. □

Proposition 17.3 (Conjugated source-side buffered tail forcing)**.** *Assume the hypotheses of Definition 17.1. Then for every sufficiently small $h > 0$ there exists a bounded operator*

$$\mathfrak{Q}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}} \in \mathcal{L}(F_{h,R}, E_{R_0})$$

such that for every $\Theta \in F_{h,R}$,

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}_-^\lambda (\mathfrak{W}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}} \Theta) = \mathfrak{Q}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}} \Theta \qquad \text{in } E_{R_0}. \tag{204}$$

Moreover, there exist $C_{t,R_0} < \infty$ and $C_{t,R_0,\alpha} < \infty$ such that

$$\left\| \mathfrak{Q}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}} \right\|_{\mathcal{L}(F_{h,R}, E_{R_0})} \le C_{t,R_0}(R-R_0)^{-\beta} + C_{t,R_0,\alpha} h^{\frac{1+\alpha}{2}} \tag{205}$$

for all sufficiently small $h > 0$.

Proof. For $0 \le r \le R_0$ and $q \in [R, \varepsilon_{h,\alpha}^{-1}]$, define

$$a_{XY,t,h}(r,q) := L_{XY}(t-hr, t-hq).$$

Exactly as in the proof of Proposition 15.5, but with fixed upper endpoint q, one obtains

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^\lambda\Big((q-\cdot)^{\mu-1}a_{XY,t,h}(\cdot,q)\Big)(r) = K_{XY,t,h}^{\mathrm{tail}}(r,q), \qquad 0 \le r \le R_0,$$

where

$$K_{XY,t,h}^{\mathrm{tail}}(r,q) := \beta(q-r)^{-\beta-1}g_{t,h}^{\mathrm{tail}}(r,q) - (q-r)^{-\beta}\partial_1 g_{t,h}^{\mathrm{tail}}(r,q)$$

and

$$g_{t,h}^{\mathrm{tail}}(r,q) := \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\mu-1}a_{XY,t,h}(r+(q-r)\theta, q)\,d\theta.$$

Therefore Fubini's theorem yields

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^\lambda(\mathfrak{W}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}}\Theta)(r) = \int_R^{\varepsilon_{h,\alpha}^{-1}} K_{XY,t,h}^{\mathrm{tail}}(r,q)\Theta(q)\,dq$$

for almost every $r \in [0, R_0]$. This defines the bounded operator

$$(\mathfrak{Q}_{XY,t,h,\alpha,R_0,R}^{\mathrm{tail}}\Theta)(r) := \int_R^{\varepsilon_{h,\alpha}^{-1}} K_{XY,t,h}^{\mathrm{tail}}(r,q)\Theta(q)\,dq, \qquad 0 \le r \le R_0,$$

and proves (204).

Since L_{XY} is C^1, one has

$$|g_{t,h}^{\mathrm{tail}}(r,q)| \le C_t, \qquad |\partial_1 g_{t,h}^{\mathrm{tail}}(r,q)| \le C_t h$$

uniformly for $0 \le r \le R_0$ and $R \le q \le \varepsilon_{h,\alpha}^{-1}$. Accordingly, write

$$K_{XY,t,h}^{\mathrm{tail}} = K_{XY,t,h}^{(1)} + K_{XY,t,h}^{(2)},$$

with

$$|K_{XY,t,h}^{(1)}(r,q)| \le C_t(q-r)^{-\beta-1}, \qquad |K_{XY,t,h}^{(2)}(r,q)| \le C_t h(q-r)^{-\beta}.$$

For $K_{XY,t,h}^{(1)}$, Schur's test gives

$$\sup_{0\le r\le R_0}\int_R^{\varepsilon_{h,\alpha}^{-1}}(q-r)^{-\beta-1}\,dq \le \frac{1}{\beta}(R-R_0)^{-\beta}$$

and

$$\sup_{R\le q\le \varepsilon_{h,\alpha}^{-1}}\int_0^{R_0}(q-r)^{-\beta-1}\,dr \le \frac{1}{\beta}(R-R_0)^{-\beta}.$$

Hence

$$\left\|K_{XY,t,h}^{(1)}\right\|_{\mathcal{L}(F_{h,R},E_{R_0})} \le C_{t,R_0}(R-R_0)^{-\beta}. \tag{206}$$

For $K_{XY,t,h}^{(2)}$, use the Hilbert–Schmidt bound:

$$\begin{aligned}\left\|K_{XY,t,h}^{(2)}\right\|^2_{L^2([0,R_0]\times[R,\varepsilon_{h,\alpha}^{-1}])} &\le C_{t,R_0}h^2\int_0^{R_0}\int_R^{\varepsilon_{h,\alpha}^{-1}}(q-r)^{-2\beta}\,dq\,dr\\ &\le C_{t,R_0,\alpha}h^2\varepsilon_{h,\alpha}^{-(1-2\beta)},\end{aligned}$$

because $2\beta < 1$. Since $\varepsilon_{h,\alpha} = h^{1-\alpha}$,

$$h\,\varepsilon_{h,\alpha}^{-\frac{1-2\beta}{2}} = h^{1-\frac{(1-\alpha)(1-2\beta)}{2}} \le h^{\frac{1+\alpha}{2}} \qquad \text{for } 0 < h \le 1.$$

Therefore

$$\left\|K_{XY,t,h}^{(2)}\right\|_{\mathcal{L}(F_{h,R},E_{R_0})} \le C_{t,R_0,\alpha}h^{\frac{1+\alpha}{2}}. \tag{207}$$

Combining (206) and (207) proves (205). $\square$

Theorem 17.4 (Buffered truncation defect for a global exact micro pair)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Let $0 < R_0 < R$, choose $h > 0$ sufficiently small that $hR < t$ and $R < \varepsilon_{h,\alpha}^{-1}$, and let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Define

$$\Phi^{\le R} := \Phi\mathbf{1}_{[0,R]}, \qquad \Phi^{>R} := \Phi\mathbf{1}_{(R,\varepsilon_{h,\alpha}^{-1}]}, \qquad \Psi^{\le R} := \Psi\mathbf{1}_{[0,R]}, \qquad \Psi^{>R} := \Psi\mathbf{1}_{(R,\varepsilon_{h,\alpha}^{-1}]}.$$

Then:

(i) the windowed truncation defect satisfies

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}\left(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi^{\le R} - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi^{\le R}\right) = \mathfrak{Q}^{\mathrm{tail}}_{XY,t,h,\alpha,R_0,R}\Phi^{>R} - \mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\Psi^{>R} \tag{208}$$

in E_{R_0};

(ii) there exists $C_{t,R_0,\alpha} < \infty$ such that

$$\begin{aligned} &\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}\left(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi^{\le R} - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi^{\le R}\right)\right\|_{E_{R_0}} \\ &\le C_{t,R_0,\alpha}\left((R-R_0)^{-\beta}\|\Phi^{>R}\|_{L^2(\mathbb{R}_+)} + h^{\frac{1+\alpha}{2}}\left(\|\Phi^{>R}\|_{L^2(\mathbb{R}_+)} + \|\Psi^{>R}\|_{L^2(\mathbb{R}_+)}\right)\right). \end{aligned} \tag{209}$$

Proof. Because $\Psi = \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi$, Corollary 14.6 gives

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi \qquad \text{in } L^2(\mathbb{R}_+).$$

Set $\varepsilon := \varepsilon_{h,\alpha}$. By Proposition 14.9(ii),

$$\Psi = h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon}\Phi.$$

Since $\mathfrak{p}_{\varepsilon}\Phi$ is supported in $[0,\varepsilon^{-1}]$, the lift $\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon}\Phi$ is supported in $[t-h\varepsilon^{-1},t] = [t-h^{\alpha},t]$. Corollary 16.1 therefore implies

$$\operatorname{supp}\bigl(\mathcal{S}_t\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon}\Phi\bigr) \subset [t-h^{\alpha},t],$$

and hence

$$\operatorname{supp}(\Psi) \subset [0,\varepsilon^{-1}].$$

Splitting the exact Volterra identity at $q = R$ therefore gives, in E_{R_0},

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi^{\le R} - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi^{\le R} = \mathfrak{W}^{\mathrm{tail}}_{XY,t,h,\alpha,R_0,R}\Phi^{>R} - \mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\Psi^{>R}.$$

Applying Propositions 17.2 and 17.3 proves (208). The norm estimate (209) follows by applying (203) and (205) to the two tail terms and then using the triangle inequality. □

18. Intrinsic exact fixed-window solver

The fixed-window graph spaces from Section 16 were defined by restricting the global micro transfer to compactly supported inputs. The exact finite-window problem can also be formulated intrinsically. The windowed Volterra relation determines a unique closed solver, and that intrinsic solver coincides with the supported-global graph.

Definition 18.1 (Intrinsic exact fixed-window graph)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, $\alpha \in (0,1)$, $R > 0$, and choose $h > 0$ such that

$$hR < t, \qquad R < \varepsilon_{h,\alpha}^{-1}.$$

Define the intrinsic exact fixed-window graph by

$$\Gamma^{\mathrm{ex}}_{t,h,\alpha,R} := \left\{(\Phi,\Psi) \in E_R \times E_R : \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi\right\}.$$

Proposition 18.2 (Closed exact fixed-window solver)**.** *Assume the hypotheses of Definition 18.1, and suppose in addition that $h > 0$ is small enough for Corollary 15.4 to apply on the window $[0, R]$. Then:*

(i) $\Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$ *is a closed linear subspace of* $E_R \times E_R$*;*

(ii) the operator $\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R} : E_R \to E_R$ *is injective;*

(iii) there exists a unique closed linear operator

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R} : \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}) \subset E_R \to E_R$$

whose graph is $\Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$.

Proof. The map

$$(\Phi, \Psi) \longmapsto \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi$$

is bounded from $E_R \times E_R$ to E_R, because both fixed-window Volterra operators are bounded. Therefore $\Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$, being its kernel, is a closed linear subspace. This proves (i).

To prove injectivity, let $\Psi \in E_R$ satisfy

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = 0.$$

Since Ψ is supported in $[0, R]$, Proposition 15.3 gives

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}\left(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi\right) = 0 \qquad \text{in } E_R,$$

where $\lambda = H_Y + \frac{1}{2}$. Corollary 15.4 implies $\Psi = 0$. This proves (ii).

Part (iii) follows from (ii): for each $\Phi \in E_R$ there is at most one $\Psi \in E_R$ such that $(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$. Hence the closed subspace $\Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$ is the graph of a unique closed linear operator. □

Proposition 18.3 (Intrinsic and supported-global exact graphs coincide)**.** *Assume the hypotheses of Proposition 18.2. Then*

$$\mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}) = \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}, \qquad \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi = \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi \quad \textit{for every } \Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}. \tag{210}$$

In particular, the graph of $\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}$ *is exactly*

$$\Gamma^{\mathrm{mic}}_{t,h,\alpha,R} := \left\{\left(\Phi, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\right) : \Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}\right\} \subset E_R \times E_R.$$

Proof. Let $\Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}$, and set

$$\Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

By Proposition 16.6(i), $\Psi \in E_R$. By Proposition 16.6(ii),

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi \qquad \text{in } E_R.$$

Hence $(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$, so

$$\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R} \subset \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R})$$

and

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi = \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi \qquad \text{for every } \Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R}.$$

Conversely, let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R})$$

and set

$$\Psi := \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi \in E_R.$$

Then

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi \qquad \text{in } E_R. \tag{211}$$

Since Φ and Ψ are supported in $[0, R] \subset [0, \varepsilon_{h,\alpha}^{-1}]$, one has

$$\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi = \Phi, \qquad \mathfrak{p}_{\varepsilon_{h,\alpha}}\Psi = \Psi.$$

Set $\varepsilon := \varepsilon_{h,\alpha}$. By Proposition 14.5,

$$\mathfrak{d}_\varepsilon \mathfrak{A}_{Y,t,h,\alpha}\mathfrak{l}_\varepsilon\Psi = \varepsilon^{H_Y}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi, \qquad \mathfrak{d}_\varepsilon \mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon\Phi = \varepsilon^{H_{XY}}\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi.$$

Multiplying (211) by ε^{H_Y}, using $\beta = H_Y - H_{XY}$, and then applying $\mathfrak{l}_\varepsilon$ together with $\mathfrak{l}_\varepsilon \mathfrak{d}_\varepsilon = I_E$ from Lemma 14.2, yields

$$\mathfrak{A}_{Y,t,h,\alpha}\big(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi\big) = \mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon \Phi. \tag{212}$$

Now define

$$w_h := \mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_\varepsilon \Phi \in H^{\text{near}}_{t,h,\alpha}, \qquad u_h := h^{-\alpha\beta}\mathcal{L}^{(\alpha)}_{t,h}\big(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi\big) \in H^{\text{near}}_{t,h,\alpha}.$$

Using Proposition 11.6 and (212),

$$\begin{aligned}
\mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{Y,t}u_h &= h^{-\alpha\beta+\alpha H_Y}\mathfrak{A}_{Y,t,h,\alpha}\big(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi\big)\\
&= h^{\alpha H_{XY}}\mathfrak{A}_{XY,t,h,\alpha}\mathfrak{l}_\varepsilon \Phi\\
&= \mathcal{D}^{(\alpha)}_{t,h}\mathcal{K}_{XY,t}w_h.
\end{aligned}$$

Both $\mathcal{K}_{Y,t}u_h$ and $\mathcal{K}_{XY,t}w_h$ are supported in $[t-h^\alpha, t]$, so Lemma 11.2 implies

$$\mathcal{K}_{Y,t}u_h = \mathcal{K}_{XY,t}w_h.$$

Hence $w_h \in \mathcal{D}(\mathcal{S}_t)$, $\mathcal{S}_t w_h = u_h$, and

$$\mathfrak{l}_\varepsilon \Phi = \mathcal{D}^{(\alpha)}_{t,h}w_h \in \mathfrak{G}^{\text{near}}_{t,h,\alpha}.$$

Therefore $\Phi \in \mathfrak{G}^{\text{mic}}_{t,h,\alpha,R}$. Moreover, by Definitions 11.7 and 14.8,

$$\begin{aligned}
\mathfrak{T}^{\text{mic}}_{t,h,\alpha}\Phi &= \varepsilon^{\beta}\mathfrak{d}_\varepsilon \mathfrak{T}_{t,h,\alpha}\mathfrak{l}_\varepsilon \Phi\\
&= \varepsilon^{\beta}\mathfrak{d}_\varepsilon\big(\varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi\big)\\
&= \mathfrak{d}_\varepsilon \mathfrak{l}_\varepsilon \Psi = \mathfrak{p}_\varepsilon \Psi = \Psi,
\end{aligned}$$

because $\operatorname{supp}\Psi \subset [0,R] \subset [0,\varepsilon^{-1}]$. This proves the reverse inclusion and completes (210). ☐

Corollary 18.4 (Intrinsic exact fixed-window graph norm)**.** *Under the hypotheses of Proposition 18.3, define*

$$\|\Phi\|^2_{\mathfrak{G}^{\text{ex}}_{t,h,\alpha,R}} := \|\Phi\|^2_{E_R} + \frac{\eta^2}{\nu_Y^2}\|\mathfrak{T}^{\text{ex}}_{t,h,\alpha,R}\Phi\|^2_{E_R}, \qquad \Phi \in \mathcal{D}(\mathfrak{T}^{\text{ex}}_{t,h,\alpha,R}).$$

Then $\mathcal{D}(\mathfrak{T}^{\text{ex}}_{t,h,\alpha,R})$, equipped with this norm, is a Hilbert space, and the identification in (210) *is isometric:*

$$\|\Phi\|_{\mathfrak{G}^{\text{ex}}_{t,h,\alpha,R}} = \|\Phi\|_{\mathfrak{G}^{\text{mic}}_{t,h,\alpha,R}} \qquad \textit{for every } \Phi \in \mathfrak{G}^{\text{mic}}_{t,h,\alpha,R}.$$

Proof. Since $\mathfrak{T}^{\text{ex}}_{t,h,\alpha,R}$ is closed by Proposition 18.2, its graph norm defines a Hilbert space. The isometric identification is exactly Proposition 18.3 together with Definition 16.5. ☐

19. Exact fixed-window regularity

The intrinsic windowed solver requires a domain description. This is the only point where the argument uses an additional derivative of the slowly varying amplitudes. With that extra regularity, the source-side conjugated operator splits into a diagonal multiplier times the fractional model derivative plus a bounded lower-order remainder.

Definition 19.1 (Extra amplitude regularity on a fixed time horizon)**.** We say that Assumption 3.1 holds with *second-order amplitude regularity* if each

$$L_i, \qquad i \in \{Y, XY\},$$

extends to a C^2 function on a neighbourhood of Δ_T.

Lemma 19.2 (Distributional model derivative on a fixed window)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

fix $t \in (0,T)$, and let $R > 0$. For $\Phi \in E_R = L^2([0,R])$, define

$$(\mathfrak{J}_{\beta,R}\Phi)(r) := \int_r^R (q-r)^{-\beta}\Phi(q)\,dq, \qquad 0 \le r \le R,$$

and let

$$\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi := -\frac{d}{dr}\mathfrak{J}_{\beta,R}\Phi$$

in $\mathcal{D}'((0,R))$. *Then:*

(i) $\Phi \in H^{\beta}((0,R))$ *if and only if* $\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi \in E_R$;

(ii) *in that case,*

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = \frac{\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi \qquad \text{in } E_R, \tag{213}$$

where $\mathbf{c}_t$ *is defined in* (82).

Proof. The equivalence in item (i) is exactly the classical finite-interval Riemann–Liouville characterization of $H^{\beta}((0,R))$ by the right-sided derivative, i.e. the rescaled version of the analytic input invoked in Theorem 12.15. In particular,

$$H^{\beta}((0,R)) = \left\{\Phi \in E_R : \mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi \in E_R\right\},$$

and the map $\Phi \mapsto \mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi$ is bounded from $H^{\beta}((0,R))$ to E_R.

For $\Phi \in C_c^{\infty}((0,R))$, Definition 8.1 and the proof of Theorem 16.3 give

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = \mathbf{c}_t\mathfrak{D}^{\beta}_{-}\Phi = -\frac{\mathbf{c}_t}{\Gamma(1-\beta)}\frac{d}{dr}\int_r^R (q-r)^{-\beta}\Phi(q)\,dq,$$

which is exactly (213) on the smooth core. Since $C_c^{\infty}((0,R))$ is dense in $H^{\beta}((0,R))$ by Theorem 16.3(iv), $\mathfrak{T}^{\mathrm{hl}}_{t,R}$ is closed with domain $H^{\beta}((0,R))$, and $\Phi \mapsto \mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi$ is bounded on $H^{\beta}((0,R))$, the identity extends to every $\Phi \in H^{\beta}((0,R))$. This proves item (ii). $\square$

Proposition 19.3 (Source-side conjugated decomposition on a fixed window)**.** *Assume Assumption 3.1 with the extra regularity from Definition 19.1. Fix* $t \in (0,T)$, $\alpha \in (0,1)$, $R > 0$, *and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

and define on $\Delta_R := \{0 \le r \le q \le R\}$

$$g^{\mathrm{win}}_{t,h,R}(r,q) := \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\mu-1}L_{XY}(t-h(r+(q-r)\theta), t-hq)\,d\theta.$$

Also define

$$g^{\mathrm{hl}}_t := \frac{L_{XY}(t,t)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(1-\beta)} = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}.$$

Then there exist $h_0 \in (0,t]$, $C_{t,R} < \infty$, *a multiplier*

$$\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R} \in \mathcal{L}(E_R), \qquad (\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\Phi)(r) := m_{t,h,\alpha,R}(r)\Phi(r),$$

and a bounded remainder operator

$$\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R} \in \mathcal{L}(E_R),$$

such that for every $0 < h \le h_0$:

(i)

$$\|m_{t,h,\alpha,R} - 1\|_{C^1([0,R])} \le C_{t,R}h; \tag{214}$$

(ii)

$$\|\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\|_{\mathcal{L}(E_R)} \le C_{t,R}h; \tag{215}$$

(iii) *for every* $\Phi \in C_c^{\infty}((0,R))$,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi) = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \qquad \text{in } E_R. \tag{216}$$

(iv) for every $\Phi \in E_R$,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^{\lambda}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi) = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \qquad \text{in } \mathcal{D}'((0,R)), \tag{217}$$

Here the left-hand side denotes the distributional derivative of $\mathfrak{I}_-^{1-\lambda}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi)$.

Proof. Because L_{XY} is C^2, the function $g^{\mathrm{win}}_{t,h,R}$ is C^2 on Δ_R, and

$$\|g^{\mathrm{win}}_{t,h,R} - g^{\mathrm{hl}}_t\|_{C^2(\Delta_R)} \le C_{t,R}h \tag{218}$$

for all sufficiently small $h > 0$. Define

$$m_{t,h,\alpha,R}(r) := \frac{g^{\mathrm{win}}_{t,h,R}(r,r)}{g^{\mathrm{hl}}_t}, \qquad 0 \le r \le R.$$

Then (214) follows from (218).

For $0 \le r < q \le R$, set

$$b_{t,h,\alpha,R}(r,q) := \frac{g^{\mathrm{win}}_{t,h,R}(r,q) - g^{\mathrm{win}}_{t,h,R}(r,r)}{q-r},$$

and extend continuously to the diagonal by

$$b_{t,h,\alpha,R}(r,r) := \partial_2 g^{\mathrm{win}}_{t,h,R}(r,r).$$

By (218), one has

$$\|b_{t,h,\alpha,R}\|_{C^1(\Delta_R)} \le C_{t,R}h. \tag{219}$$

Now let $\Phi \in C_c^\infty((0,R))$. The proof of Proposition 15.5 gives

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^{\lambda}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi)(r) = -\frac{d}{dr}\int_r^R (q-r)^{-\beta} g^{\mathrm{win}}_{t,h,R}(r,q)\Phi(q)\,dq$$

for almost every $r \in [0,R]$. Decomposing

$$g^{\mathrm{win}}_{t,h,R}(r,q) = g^{\mathrm{win}}_{t,h,R}(r,r) + (q-r)b_{t,h,\alpha,R}(r,q),$$

we obtain

$$\begin{aligned}\frac{1}{\Gamma(\lambda)}\mathfrak{D}_-^{\lambda}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi)(r) = {}& -g^{\mathrm{win}}_{t,h,R}(r,r)\frac{d}{dr}\int_r^R (q-r)^{-\beta}\Phi(q)\,dq \\ & - (\partial_1 g^{\mathrm{win}}_{t,h,R})(r,r)\int_r^R (q-r)^{-\beta}\Phi(q)\,dq \\ & - \frac{d}{dr}\int_r^R (q-r)^{1-\beta} b_{t,h,\alpha,R}(r,q)\Phi(q)\,dq.\end{aligned}$$

Since

$$g^{\mathrm{win}}_{t,h,R}(r,r) = g^{\mathrm{hl}}_t m_{t,h,\alpha,R}(r) = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)} m_{t,h,\alpha,R}(r),$$

Lemma 19.2 yields

$$-g^{\mathrm{win}}_{t,h,R}(r,r)\frac{d}{dr}\int_r^R (q-r)^{-\beta}\Phi(q)\,dq = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi(r).$$

Define $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}$ by

$$\begin{aligned}(\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi)(r) := {}& -(\partial_1 g^{\mathrm{win}}_{t,h,R})(r,r)\int_r^R (q-r)^{-\beta}\Phi(q)\,dq \\ & - \frac{d}{dr}\int_r^R (q-r)^{1-\beta} b_{t,h,\alpha,R}(r,q)\Phi(q)\,dq.\end{aligned}$$

This proves (216) on the smooth core.

It remains to show that $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}$ extends boundedly to E_R with norm $O(h)$. For the first term, (218) and the Hilbert–Schmidt property of the kernel $(q-r)^{-\beta}\mathbf{1}_{\Delta_R}$ give an $O(h)$ operator bound.

For the second term, the lower endpoint contributes no boundary term because $1-\beta>0$, so differentiation under the integral sign yields

$$-\frac{d}{dr}\int_r^R (q-r)^{1-\beta} b_{t,h,\alpha,R}(r,q)\Phi(q)\,dq = \int_r^R K^{\mathrm{rem}}_{t,h,\alpha,R}(r,q)\Phi(q)\,dq,$$

where

$$K^{\mathrm{rem}}_{t,h,\alpha,R}(r,q) = -(1-\beta)(q-r)^{-\beta} b_{t,h,\alpha,R}(r,q) - (q-r)^{1-\beta}\partial_1 b_{t,h,\alpha,R}(r,q).$$

By (219),

$$|K^{\mathrm{rem}}_{t,h,\alpha,R}(r,q)| \le C_{t,R}h\big((q-r)^{-\beta} + (q-r)^{1-\beta}\big)\mathbf{1}_{\Delta_R}(r,q),$$

and the right-hand side is Hilbert–Schmidt on Δ_R because $\beta<\frac{1}{2}$. Hence (215) follows.

To prove (217), first note that the same Fubini computation as above shows that for every $\Phi\in E_R$,

$$(\mathfrak{I}^{1-\lambda}_{-}\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi)(r) = \int_r^R (q-r)^{-\beta} g^{\mathrm{win}}_{t,h,R}(r,q)\Phi(q)\,dq, \qquad 0\le r\le R.$$

Because the kernel $(q-r)^{-\beta} g^{\mathrm{win}}_{t,h,R}(r,q)\mathbf{1}_{\Delta_R}(r,q)$ is square-integrable on Δ_R, the map

$$\Phi \longmapsto \mathfrak{I}^{1-\lambda}_{-}\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi$$

is bounded from E_R to E_R, hence the left-hand side of (217) defines a continuous linear map $E_R\to\mathcal{D}'((0,R))$ after distributional differentiation. By the boundedness of $\mathfrak{J}_{\beta,R}$ on E_R, the definition of $\mathfrak{D}^{\mathrm{dist}}_{\beta,R}$, and (215), the right-hand side of (217) is also continuous from E_R to $\mathcal{D}'((0,R))$. Since $C_c^\infty((0,R))$ is dense in E_R and (217) coincides with (216) on the smooth core by Lemma 19.2, the distributional identity extends to every $\Phi\in E_R$. ☐

Theorem 19.4 (Fractional Sobolev characterization of the exact fixed-window solver)**.** *Assume Assumptions 3.1 and 4.1, and suppose that Definition 19.1 holds. Let $\eta>0$, fix $t\in(0,T)$, $\alpha\in(0,1)$, and $R>0$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then there exist $h_0\in(0,t]$ and $C_{t,R}<\infty$ such that for every $0<h\le h_0$ with

$$hR<t, \qquad R<\varepsilon^{-1}_{h,\alpha},$$

the following hold:

(i)

$$\mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}) = H^\beta((0,R)); \tag{220}$$

(ii) there exist bounded operators

$$\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}, \mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R} \in \mathcal{L}(E_R)$$

such that $\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}$ is invertible,

$$\left\|\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R} - I_{E_R}\right\|_{\mathcal{L}(E_R)} + \left\|\mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R}\right\|_{\mathcal{L}(E_R)} \le C_{t,R}h, \tag{221}$$

and

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi = \mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \qquad \textit{for every } \Phi\in H^\beta((0,R)); \tag{222}$$

(iii) the exact graph norm is uniformly equivalent to the H^β norm:

$$C^{-1}_{t,R}\|\Phi\|_{H^\beta((0,R))} \le \|\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R}} \le C_{t,R}\|\Phi\|_{H^\beta((0,R))} \qquad \textit{for every } \Phi\in H^\beta((0,R)). \tag{223}$$

Proof. Let h_0 be small enough that Proposition 19.3 and Corollary 15.4 both hold for every $0<h\le h_0$. For such h, define

$$\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R} := (\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R})^{-1}L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}, \qquad \mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R} := (\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R})^{-1}\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}.$$

By (183), (184), (214), and (215), one has

$$\left\|\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R} - I_{E_R}\right\|_{\mathcal{L}(E_R)} \le C_{t,R}h, \qquad \left\|\mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R}\right\|_{\mathcal{L}(E_R)} \le C_{t,R}h,$$

after possibly shrinking h_0. In particular $\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}$ is invertible with uniformly bounded inverse, which proves (221).

Now let $\Phi \in C_c^\infty((0,R))$, and define

$$\Psi_h := \mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \in E_R.$$

By construction and Proposition 19.3,

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi_h = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi = \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi)$$

in E_R. Exactly as in the proof of Theorem 15.6, both

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi_h \quad \text{and} \quad \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi$$

are compactly supported in $[0,R]$, have the same conjugated λ-derivative, and therefore coincide. Hence

$$(\Phi,\Psi_h) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R},$$

so by Proposition 18.2,

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi = \Psi_h \qquad \text{for every } \Phi \in C_c^\infty((0,R)).$$

Thus (222) holds on the smooth core.

Since $C_c^\infty((0,R))$ is dense in $H^\beta((0,R))$ and $\mathfrak{T}^{\mathrm{hl}}_{t,R}$ is closed with domain $H^\beta((0,R))$, (222) defines a closed operator on $H^\beta((0,R))$. Passing to the limit along smooth approximations and using the boundedness of $\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}$ and $\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}$, we conclude that

$$H^\beta((0,R)) \subset \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R})$$

and (222) holds for every $\Phi \in H^\beta((0,R))$.

For the reverse inclusion, let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}), \qquad \Psi := \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi \in E_R.$$

Then $(\Phi,\Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$, so

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\Phi \qquad \text{in } E_R.$$

Proposition 15.3(i) and Proposition 19.3(iv) therefore give

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi$$

in $\mathcal{D}'((0,R))$. Since $\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi \in E_R$ and $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \in E_R$, the multiplier $\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}$, being boundedly invertible for small h, shows that $\mathfrak{D}^{\mathrm{dist}}_{\beta,R}\Phi \in E_R$. Lemma 19.2 then yields $\Phi \in H^\beta((0,R))$. This proves (220), and (222) follows from the uniqueness part of Proposition 18.2.

Finally, (222), (221), and the boundedness of $(\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R})^{-1}$ imply

$$\begin{aligned}\|\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi\|_{E_R} &\le C_{t,R}(\|\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi\|_{E_R} + \|\Phi\|_{E_R}),\\ \|\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi\|_{E_R} &\le C_{t,R}(\|\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi\|_{E_R} + \|\Phi\|_{E_R})\end{aligned}$$

for every $\Phi \in H^\beta((0,R))$. Combining these bounds with Theorem 16.3(iii), which identifies the model graph norm with the $H^\beta((0,R))$ norm, proves (223). $\square$

Proposition 19.5 (Conjugated characterization of the exact fixed-window graph)**.** *Assume the hypotheses of Theorem 19.4. Then for every*

$$\Phi \in H^\beta((0,R)), \qquad \Psi \in E_R,$$

the following are equivalent:

(i) $(\Phi,\Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$*;*

(ii)

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \qquad \textit{in } E_R. \tag{224}$$

Proof. If $(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$, then

$$\Psi = \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi$$

by Proposition 18.2. Applying (222) from Theorem 19.4 and then the definitions of $\mathfrak{A}^{\mathrm{win}}_{t,h,\alpha,R}$ and $\mathfrak{R}^{\mathrm{win}}_{t,h,\alpha,R}$ gives (224).

Conversely, suppose (224) holds. Since $\Phi \in H^{\beta}((0,R))$, Theorem 19.4 yields

$$\Psi_* := \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi \in E_R$$

and

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi_* = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi$$

in E_R. Subtracting from (224) gives

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}(\Psi - \Psi_*) = 0.$$

Corollary 15.4 therefore implies $\Psi = \Psi_*$. Hence

$$(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R},$$

which proves (i). □

Theorem 19.6 (Stable correction of a full-window conjugated defect)**.** *Assume the hypotheses of Theorem 19.4. Then there exists $C_{t,R} < \infty$ such that for every*

$$\Phi \in H^{\beta}((0,R)), \qquad \Psi \in E_R,$$

the conjugated defect

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi,\Psi) := \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\Psi - L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi - \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \in E_R \tag{225}$$

satisfies the exact correction formula

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi = \Psi - \left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\right)^{-1}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi,\Psi), \tag{226}$$

and the stability estimate

$$\left\|\Psi - \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi\right\|_{E_R} \le C_{t,R}\left\|\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi,\Psi)\right\|_{E_R}. \tag{227}$$

Proof. Let

$$f := \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi,\Psi) \in E_R,$$

and define

$$\widetilde{\Psi} := \Psi - \left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\right)^{-1} f \in E_R.$$

By construction,

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\widetilde{\Psi} = L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}\Phi \qquad \text{in } E_R.$$

Proposition 19.5 therefore implies

$$(\Phi, \widetilde{\Psi}) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R},$$

hence

$$\widetilde{\Psi} = \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi.$$

This proves (226). The bound (227) then follows from (184). □

Definition 19.7 (Smooth buffered cutoff on the micro half-line)**.** Let $0 < R_0 < R_1 < R_2 < \infty$. A function

$$\chi \in C^{\infty}([0,\infty))$$

is called a *smooth buffered cutoff* with radii (R_0, R_1, R_2) if

$$0 \le \chi \le 1 \quad \text{on } [0,\infty), \qquad \chi \equiv 1 \text{ on } [0, R_1], \qquad \operatorname{supp}\chi \subset [0, R_2].$$

Proposition 19.8 (Full-window commutator forcing for a smooth cutoff)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \beta := H_Y - H_{XY}.$$

Let $0 < R_1 < R_2$, let

$$\chi \in C^\infty([0,\infty)), \qquad 0 \le \chi \le 1, \qquad \operatorname{supp} \chi \subset [0, R_2],$$

and choose $h > 0$ sufficiently small that

$$hR_2 < t, \qquad R_2 < \varepsilon_{h,\alpha}^{-1}.$$

For $\Theta \in L^2(\mathbb{R}_+)$, define on $E_{R_2} = L^2([0,R_2])$ the commutator Volterra operators

$$(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta)(r) := \int_r^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\lambda-1}\big(\chi(q)-\chi(r)\big) L_Y(t-hr, t-hq)\Theta(q)\,dq, \tag{228}$$

$$(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta)(r) := \int_r^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\mu-1}\big(\chi(q)-\chi(r)\big) L_{XY}(t-hr, t-hq)\Theta(q)\,dq \tag{229}$$

for $0 \le r \le R_2$. Then there exist bounded operators

$$\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}, \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi} \in \mathcal{L}(L^2(\mathbb{R}_+), E_{R_2})$$

such that for every $\Theta \in L^2(\mathbb{R}_+)$,

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_- (\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta) = \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta \qquad \text{in } E_{R_2}, \tag{230}$$

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_- (\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta) = \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta \qquad \text{in } E_{R_2}. \tag{231}$$

Proof. Set $\varepsilon := \varepsilon_{h,\alpha}$. For $0 \le r < q \le \varepsilon^{-1}$, define

$$\delta_\chi(r,q) := \frac{\chi(q)-\chi(r)}{q-r},$$

and extend continuously to the diagonal by $\delta_\chi(r,r) := \chi'(r)$. Since χ is smooth and compactly supported in $[0,R_2]$, the function δ_χ is C^1 on

$$\Delta_{R_2,\varepsilon} := \{0 \le r \le q \le \varepsilon^{-1}\},$$

with finite C^1 norm.

For the target-side commutator, write

$$(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta)(r) = \int_r^{\varepsilon^{-1}} (q-r)^\lambda \delta_\chi(r,q) L_Y(t-hr,t-hq)\Theta(q)\,dq.$$

Exactly the same Fubini computation as in Proposition 17.2, now with exponent λ instead of $\lambda - 1$, shows that

$$\mathfrak{I}^{1-\lambda}_- (\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta)(r) = \int_r^{\varepsilon^{-1}} (q-r) F_{Y,t,h,\chi}(r,q)\Theta(q)\,dq,$$

where $F_{Y,t,h,\chi}$ is C^1 on $\Delta_{R_2,\varepsilon}$. Differentiating under the integral sign therefore yields

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}^\lambda_- (\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta)(r) = F_{Y,t,h,\chi}(r,r)\Theta(r) - \int_r^{\varepsilon^{-1}} K^{\mathrm{com}}_{Y,t,h,\chi}(r,q)\Theta(q)\,dq$$

for almost every $r \in [0,R_2]$, with bounded kernel

$$K^{\mathrm{com}}_{Y,t,h,\chi}(r,q) := F_{Y,t,h,\chi}(r,q) + (q-r)\partial_1 F_{Y,t,h,\chi}(r,q).$$

Since $\Delta_{R_2,\varepsilon}$ has finite measure, this defines a bounded operator $\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi} \in \mathcal{L}(L^2(\mathbb{R}_+), E_{R_2})$, proving (230).

For the source-side commutator, write

$$(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta)(r) = \int_r^{\varepsilon^{-1}} (q-r)^{\mu}\delta_\chi(r,q)L_{XY}(t-hr,t-hq)\Theta(q)\,dq.$$

Exactly as in Proposition 15.5, the same Beta-function computation gives

$$\mathfrak{I}^{1-\lambda}_{-}(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta)(r) = \int_r^{\varepsilon^{-1}} (q-r)^{1-\beta}F_{XY,t,h,\chi}(r,q)\Theta(q)\,dq,$$

where $F_{XY,t,h,\chi}$ is C^1 on $\Delta_{R_2,\varepsilon}$. Because $1-\beta>0$, differentiation under the integral sign yields

$$\begin{aligned}\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta)(r) &= -(1-\beta)\int_r^{\varepsilon^{-1}} (q-r)^{-\beta}F_{XY,t,h,\chi}(r,q)\Theta(q)\,dq \\ &\quad - \int_r^{\varepsilon^{-1}} (q-r)^{1-\beta}\partial_1 F_{XY,t,h,\chi}(r,q)\Theta(q)\,dq\end{aligned}$$

for almost every $r\in[0,R_2]$. Since $\beta<\frac12$ and $\Delta_{R_2,\varepsilon}$ has finite measure, both kernels are square-integrable on $\Delta_{R_2,\varepsilon}$. Hence this defines a bounded operator $\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\in\mathcal{L}(L^2(\mathbb{R}_+),E_{R_2})$, proving (231). ☐

Theorem 19.9 (Smooth buffered localization of a global exact micro pair)**.** *Assume Assumptions 3.1 and 4.1, let $\eta>0$, fix $t\in(0,T)$, $\alpha\in(0,1)$, and suppose*

$$0<H_{XY}<H_Y<\tfrac12.$$

Let $0<R_0<R_1<R_2$, let χ be a smooth buffered cutoff with radii (R_0,R_1,R_2), and choose $h>0$ sufficiently small that

$$hR_2<t, \qquad R_2<\varepsilon_{h,\alpha}^{-1}.$$

Let

$$\Phi\in\mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi:=\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Then:

(i) $\chi\Phi\in H^{\beta}((0,R_2))$, and the localized pair satisfies the full-window defect identity

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi) = \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi \qquad \text{in } E_{R_2}; \tag{232}$$

(ii) on the inner window $[0,R_0]$, the full-window defect reduces exactly to the buffered tail forcing:

$$\mathbf{1}_{[0,R_0]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi) = \mathfrak{Q}^{\mathrm{tail}}_{XY,t,h,\alpha,R_0,R_1}((1-\chi)\Phi) - \mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R_1}((1-\chi)\Psi) \qquad \text{in } E_{R_0}; \tag{233}$$

(iii) there exists $C_{t,R_0,\alpha}<\infty$ such that

$$\begin{aligned}&\left\|\mathbf{1}_{[0,R_0]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi)\right\|_{E_{R_0}} \\ &\le C_{t,R_0,\alpha}\big((R_1-R_0)^{-\beta}+h^{\frac{1+\alpha}{2}}\big)\big(\|(1-\chi)\Phi\|_{L^2(\mathbb{R}_+)}+\|(1-\chi)\Psi\|_{L^2(\mathbb{R}_+)}\big);\end{aligned} \tag{234}$$

(iv) defining

$$\Psi^{\mathrm{corr}}_{\chi} := \chi\Psi - \big(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_2}\big)^{-1}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi), \tag{235}$$

one has

$$(\chi\Phi,\Psi^{\mathrm{corr}}_{\chi})\in\Gamma^{\mathrm{ex}}_{t,h,\alpha,R_2}.$$

Proof. Set $\varepsilon:=\varepsilon_{h,\alpha}$. Since χ is supported in $[0,R_2]$, one has

$$\chi\Phi,\chi\Psi\in E_{R_2}.$$

By Corollary 14.6,

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi \qquad \text{in } L^2(\mathbb{R}_+).$$

Multiplying by $\chi(r)$ and subtracting from the Volterra operators applied to $\chi\Phi$ and $\chi\Psi$ gives, on $[0,R_2]$,

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R_2}(\chi\Psi) - \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R_2}(\chi\Phi) = \mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi - \mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi.$$

Applying Proposition 19.8 and the distributional source-side decomposition (217) from Proposition 19.3 therefore yields

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_2}(\chi\Psi) = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R_2}\mathfrak{D}^{\mathrm{dist}}_{\beta,R_2}(\chi\Phi) + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R_2}(\chi\Phi) + \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi$$

in $\mathcal{D}'((0,R_2))$. Since the rightmost two terms belong to E_{R_2}, Lemma 19.2 implies $\chi\Phi \in H^\beta((0,R_2))$. This proves the regularity claim in (i), and (232) is now exactly the definition of the defect $\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi)$.

For $0 \le r \le R_0$, one has $\chi(r) = 1$. Since $\chi \equiv 1$ on $[0,R_1]$, $(1-\chi)\Theta$ vanishes on $[0,R_1]$ for every Θ. Therefore

$$(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi)(r) = -\int_{R_1}^{\varepsilon^{-1}} (q-r)^{\lambda-1} L_Y(t-hr,t-hq)(1-\chi(q))\Psi(q)\,dq$$

and similarly

$$(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi)(r) = -\int_{R_1}^{\varepsilon^{-1}} (q-r)^{\mu-1} L_{XY}(t-hr,t-hq)(1-\chi(q))\Phi(q)\,dq.$$

Because Φ and Ψ are supported in $[0,\varepsilon^{-1}]$, the functions $(1-\chi)\Phi$ and $(1-\chi)\Psi$ belong to F_{h,R_1}. Thus, on $[0,R_0]$,

$$\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi = -\mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R_1}((1-\chi)\Psi),$$

and likewise for the source side. Applying Propositions 17.2 and 17.3 proves (233). The bound (234) follows from (203), (205), and the triangle inequality.

Finally, part (iv) is exactly Theorem 19.6 applied with Φ replaced by $\chi\Phi$ and Ψ replaced by $\chi\Psi$. □

20. Quantitative fixed-window localization

Localization needs a reusable fixed-window estimate. The smooth-buffer theorem starts from a global exact micro pair and produces an exact corrected pair on the localized window. On a finite window, the commutator created by a smooth cutoff is controlled quantitatively by the width of the transition layer. Exact fixed-window graph elements can therefore be localized with an explicit loss.

Lemma 20.1 (Quantitative smooth buffered cutoffs)**.** *There exists a universal constant $C_\chi < \infty$ such that for every*

$$0 < R_1 < R_2 < \infty$$

there exists a smooth buffered cutoff

$$\chi_{R_1,R_2} \in C^\infty([0,\infty))$$

with

$$0 \le \chi_{R_1,R_2} \le 1, \qquad \chi_{R_1,R_2} \equiv 1 \text{ on } [0,R_1], \qquad \operatorname{supp}\chi_{R_1,R_2} \subset [0,R_2],$$

and

$$\|\chi'_{R_1,R_2}\|_{L^\infty(\mathbb{R}_+)} \le \frac{C_\chi}{R_2-R_1}, \qquad \|\chi''_{R_1,R_2}\|_{L^\infty(\mathbb{R}_+)} \le \frac{C_\chi}{(R_2-R_1)^2}. \tag{236}$$

If

$$\delta_{R_1,R_2}(r,q) := \frac{\chi_{R_1,R_2}(q)-\chi_{R_1,R_2}(r)}{q-r}, \qquad 0 \le r < q \le R_2,$$

with continuous extension $\delta_{R_1,R_2}(r,r) := \chi'_{R_1,R_2}(r)$, then

$$\|\delta_{R_1,R_2}\|_{C^1(\Delta_{R_2})} \le C_\chi\left(\frac{1}{R_2-R_1} + \frac{1}{(R_2-R_1)^2}\right), \qquad \Delta_{R_2} := \{0 \le r \le q \le R_2\}. \tag{237}$$

Proof. Choose $\vartheta \in C^\infty(\mathbb{R})$ such that

$$0 \le \vartheta \le 1, \qquad \vartheta \equiv 1 \text{ on } (-\infty, 0], \qquad \vartheta \equiv 0 \text{ on } [1, \infty).$$

For $0 < R_1 < R_2$, define

$$\chi_{R_1,R_2}(r) := \vartheta\left(\frac{r - R_1}{R_2 - R_1}\right), \qquad r \ge 0.$$

Then χ_{R_1,R_2} is a smooth buffered cutoff with the required support and plateau properties, and (236) follows by the chain rule.

For $0 \le r < q \le R_2$, the mean-value theorem gives

$$|\delta_{R_1,R_2}(r,q)| \le \|\chi'_{R_1,R_2}\|_{L^\infty(\mathbb{R}_+)},$$

so the first term in (237) follows from (236). Moreover,

$$\partial_r \delta_{R_1,R_2}(r,q) = \frac{\chi_{R_1,R_2}(q) - \chi_{R_1,R_2}(r) - \chi'_{R_1,R_2}(r)(q-r)}{(q-r)^2},$$

and the analogous formula holds for $\partial_q \delta_{R_1,R_2}$. Taylor's theorem therefore implies

$$|\partial_r \delta_{R_1,R_2}(r,q)| + |\partial_q \delta_{R_1,R_2}(r,q)| \le \frac{1}{2}\|\chi''_{R_1,R_2}\|_{L^\infty(\mathbb{R}_+)},$$

which extends continuously to the diagonal. Combining this with (236) proves (237). $\square$

Proposition 20.2 (Quantitative commutator control on a fixed window)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then for every $R_2 < \infty$ there exists $C_{t,R_2} < \infty$ such that the following holds. Let χ be a smooth buffered cutoff with support in $[0, R_2]$, define

$$\delta_\chi(r,q) := \frac{\chi(q) - \chi(r)}{q - r}, \qquad \Lambda_{\chi,R_2} := \|\delta_\chi\|_{C^1(\Delta_{R_2})},$$

and choose $h > 0$ sufficiently small that

$$hR_2 < t, \qquad R_2 < \varepsilon_{h,\alpha}^{-1}.$$

Then for every $\Theta \in E_{R_2} = L^2([0,R_2])$,

$$\left\|\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} + \left\|\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_{t,R_2}\Lambda_{\chi,R_2}\|\Theta\|_{E_{R_2}}. \tag{238}$$

Proof. Let $\Theta \in E_{R_2}$. Since Θ is supported in $[0, R_2]$, the formulas in the proof of Proposition 19.8 reduce to integrals over the fixed triangle Δ_{R_2}.

On the target side, the kernel $F_{Y,t,h,\chi}$ appearing there depends linearly on δ_χ and its first derivatives, multiplied by bounded C^1 coefficients coming from $L_Y(t - hr, t - hq)$. Hence

$$\|F_{Y,t,h,\chi}\|_{C^1(\Delta_{R_2})} \le C_{t,R_2}\Lambda_{\chi,R_2},$$

and therefore the kernel $K^{\mathrm{com}}_{Y,t,h,\chi}$ from (230) satisfies

$$\|K^{\mathrm{com}}_{Y,t,h,\chi}\|_{L^\infty(\Delta_{R_2})} \le C_{t,R_2}\Lambda_{\chi,R_2}.$$

Since Δ_{R_2} has finite measure, the corresponding integral operator is bounded from E_{R_2} to itself with norm at most $C_{t,R_2}\Lambda_{\chi,R_2}$.

On the source side, the proof of Proposition 19.8 gives a conjugated kernel of the form

$$(q-r)^{-\beta}F_{XY,t,h,\chi}(r,q) + (q-r)^{1-\beta}\partial_1 F_{XY,t,h,\chi}(r,q), \qquad (r,q) \in \Delta_{R_2},$$

with

$$\|F_{XY,t,h,\chi}\|_{C^1(\Delta_{R_2})} \le C_{t,R_2}\Lambda_{\chi,R_2}.$$

Because $\beta < \frac{1}{2}$, the kernels

$$(q-r)^{-\beta}\mathbf{1}_{\Delta_{R_2}}(r,q), \qquad (q-r)^{1-\beta}\mathbf{1}_{\Delta_{R_2}}(r,q)$$

belong to $L^2(\Delta_{R_2})$. Hence the source-side conjugated commutator operator is Hilbert–Schmidt on E_{R_2}, with norm bounded by $C_{t,R_2}\Lambda_{\chi,R_2}$. Combining the two bounds proves (238). ☐

Theorem 20.3 (Quantitative smooth localization of an exact fixed-window pair)**.** *Assume the hypotheses of Theorem 19.4. Then for every $R_2 < \infty$ there exists $C_{t,R_2} < \infty$ such that the following holds whenever $h > 0$ is sufficiently small and*

$$0 < R_1 < R_2 < \varepsilon_{h,\alpha}^{-1}, \qquad hR_2 < t.$$

Let χ be a smooth buffered cutoff with support in $[0, R_2]$, and let

$$(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R_2}.$$

Define

$$\delta_\chi(r,q) := \frac{\chi(q) - \chi(r)}{q - r}, \qquad \Lambda_{\chi,R_2} := \|\delta_\chi\|_{C^1(\Delta_{R_2})}.$$

Then:

(i) $\chi\Phi \in H^\beta((0,R_2))$, and the localized pair satisfies

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi, \chi\Psi) = \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi \qquad \text{in } E_{R_2}; \tag{239}$$

(ii) the corrected output

$$\Psi^{\mathrm{ex}}_\chi := \chi\Psi - \left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R_2}\right)^{-1}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi, \chi\Psi) \tag{240}$$

satisfies

$$(\chi\Phi, \Psi^{\mathrm{ex}}_\chi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R_2};$$

(iii) one has the quantitative correction bound

$$\left\|\Psi^{\mathrm{ex}}_\chi - \chi\Psi\right\|_{E_{R_2}} \le C_{t,R_2}\Lambda_{\chi,R_2}\left(\|\Phi\|_{E_{R_2}} + \|\Psi\|_{E_{R_2}}\right); \tag{241}$$

(iv) consequently,

$$\|\chi\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R_2}} \le C_{t,R_2}(1 + \Lambda_{\chi,R_2})\|\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R_2}}. \tag{242}$$

If $\chi = \chi_{R_1,R_2}$ is chosen as in Lemma 20.1, then

$$\left\|\Psi^{\mathrm{ex}}_{\chi_{R_1,R_2}} - \chi_{R_1,R_2}\Psi\right\|_{E_{R_2}} \le C_{t,R_2}\left(\frac{1}{R_2 - R_1} + \frac{1}{(R_2 - R_1)^2}\right)\|\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R_2}}. \tag{243}$$

Proof. By Proposition 18.3,

$$\Phi \in \mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha,R_2}, \qquad \Psi = \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi,$$

with both Φ and Ψ supported in $[0, R_2]$. Repeating the proof of Theorem 19.9(i), but now with no outer tail contribution because $\Phi, \Psi \in E_{R_2}$, yields $\chi\Phi \in H^\beta((0,R_2))$ together with the defect identity (239).

Part (ii) is exactly Theorem 19.6 applied to the pair $(\chi\Phi, \chi\Psi)$. For part (iii), combine the correction estimate (227) with (239) and then invoke Proposition 20.2:

$$\begin{aligned}\left\|\Psi^{\mathrm{ex}}_\chi - \chi\Psi\right\|_{E_{R_2}} &\le C_{t,R_2}\left\|\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi, \chi\Psi)\right\|_{E_{R_2}} \\ &\le C_{t,R_2}\Lambda_{\chi,R_2}\left(\|\Phi\|_{E_{R_2}} + \|\Psi\|_{E_{R_2}}\right).\end{aligned}$$

This proves (241).

Finally,

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi) = \Psi^{\mathrm{ex}}_\chi$$

by part (ii), so using $|\chi| \le 1$ and (241) gives

$$\begin{aligned}\|\chi\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R_2}} &\le \|\Phi\|_{E_{R_2}} + \frac{\eta}{\nu_Y}\left(\|\chi\Psi\|_{E_{R_2}} + \|\Psi^{\mathrm{ex}}_\chi - \chi\Psi\|_{E_{R_2}}\right) \\ &\le C_{t,R_2}(1 + \Lambda_{\chi,R_2})\left(\|\Phi\|_{E_{R_2}} + \frac{\eta}{\nu_Y}\|\Psi\|_{E_{R_2}}\right),\end{aligned}$$

which is exactly (242).

If $\chi = \chi_{R_1,R_2}$ is the cutoff from Lemma 20.1, then (243) follows from (241), (237), and the elementary bound

$$\|\Phi\|_{E_{R_2}} + \|\Psi\|_{E_{R_2}} \le C_{\eta,\nu_Y} \|\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R_2}}. \qquad \square$$

21. Fixed-window dual collapse and abstract global reduction

The local estimates reduce global collapse to a localization passage from the global micro graph to exact fixed windows. Two consequences have already been proved:

(i) on every fixed window, the exact micro dual norm collapses to 0 as $h \downarrow 0$;
(ii) any global localization principle reducing graph-unit micro sequences to fixed windows forces the full observable micro dual norm to collapse as well.

Definition 21.1 (Global and fixed-window exact micro functionals)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0, T)$, $\alpha \in (0, 1)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

For

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}),$$

define the global micro graph norm

$$\|\Phi\|^2_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} := \|\Phi\|^2_{L^2(\mathbb{R}_+)} + \frac{\eta^2}{\nu_Y^2} \left\|\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\right\|^2_{L^2(\mathbb{R}_+)}, \tag{244}$$

and the global exact micro functional

$$\Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi) := \left\langle \mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}, \Phi \right\rangle_{L^2(\mathbb{R}_+)} - \left\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha}, \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi \right\rangle_{L^2(\mathbb{R}_+)}. \tag{245}$$

For $R > 0$ with $hR < t$ and $R < \varepsilon^{-1}_{h,\alpha}$, and

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}),$$

define the fixed-window exact functional

$$\Lambda^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi) := \left\langle \mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}, \Phi \right\rangle_{L^2(\mathbb{R}_+)} - \left\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha}, \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}\Phi \right\rangle_{L^2(\mathbb{R}_+)}. \tag{246}$$

Finally, define the corresponding dual norms

$$\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} := \sup\left\{ \left|\Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi)\right| : \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}),\ \|\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} \le 1 \right\}, \tag{247}$$

$$\mathfrak{D}^{\mathrm{ex}}_{t,h,\alpha,R} := \sup\left\{ \left|\Lambda^{\mathrm{ex}}_{t,h,\alpha,R}(\Phi)\right| : \Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R}),\ \|\Phi\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R}} \le 1 \right\}. \tag{248}$$

Proposition 21.2 (Continuity of the global and fixed-window exact functionals)**.** *Assume the hypotheses of Definition 21.1. Then:*

(i) $\Lambda^{\mathrm{mic}}_{t,h,\alpha}$ *is continuous on* $\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}$*;*
(ii) if $R > 0$ *with* $hR < t$ *and* $R < \varepsilon^{-1}_{h,\alpha}$*, then* $\Lambda^{\mathrm{ex}}_{t,h,\alpha,R}$ *is continuous on* $\mathfrak{G}^{\mathrm{ex}}_{t,h,\alpha,R}$*.*

Proof. Since $\mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}, \mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha} \in L^2(\mathbb{R}_+)$, Cauchy–Schwarz gives

$$\left|\Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi)\right| \le \|\mathfrak{g}^{\mathrm{mic}}_{XY,t,h,\alpha}\|_{L^2} \|\Phi\|_{L^2} + \|\mathfrak{g}^{\mathrm{mic}}_{Y,t,h,\alpha}\|_{L^2} \left\|\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\right\|_{L^2},$$

which proves item (i) via (244). The fixed-window statement is identical, using (246) and Corollary 18.4. $\square$

Theorem 21.3 (Fixed-window exact dual norms collapse)**.** *Assume Assumptions 3.1 and 4.1, let* $\eta > 0$*, fix* $t \in (0, T)$*,* $\alpha \in (0, 1)$*,* $R > 0$*, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Then

$$\mathfrak{D}^{\mathrm{ex}}_{t,h,\alpha,R} \longrightarrow 0 \qquad \text{as } h \downarrow 0 \tag{249}$$

along every sequence with $hR < t$ *and* $R < \varepsilon^{-1}_{h,\alpha}$*.*

Proof. Suppose the claim fails. Then there exist $\epsilon_0 > 0$, a sequence $h_n \downarrow 0$ with

$$h_n R < t, \qquad R < \varepsilon_{h_n,\alpha}^{-1},$$

and elements

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h_n,\alpha,R})$$

such that

$$\|\Phi_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R}} \le 1, \qquad \left|\Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\Phi_n)\right| \ge \epsilon_0.$$

By Corollary 18.4 and Proposition 18.3, we may regard Φ_n as elements of $\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha,R}$ with the same norm bound. Proposition 16.8 therefore yields a subsequence, still denoted (Φ_n), and some

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{hl}}_{t,R})$$

such that

$$\Phi_n \rightharpoonup \Phi \qquad \text{weakly in } E_R, \tag{250}$$

$$\mathfrak{T}^{\mathrm{ex}}_{t,h_n,\alpha,R}\Phi_n \rightharpoonup \mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi \qquad \text{weakly in } E_R. \tag{251}$$

By Theorem 14.10,

$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h_n,\alpha} \longrightarrow L_i(t,t)\psi_{H_i} \qquad \text{strongly in } L^2(\mathbb{R}_+)$$

for $i \in \{Y, XY\}$. Since Φ_n and $\mathfrak{T}^{\mathrm{ex}}_{t,h_n,\alpha,R}\Phi_n$ are supported in $[0,R]$, the pairings in (246) may be taken over E_R. Combining the strong profile convergence with (250)–(251) gives

$$\Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\Phi_n) \longrightarrow L_{XY}(t,t)\langle \psi_{H_{XY}}, \Phi\rangle_{L^2(\mathbb{R}_+)} - L_Y(t,t)\left\langle \psi_{H_Y}, \mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi\right\rangle_{L^2(\mathbb{R}_+)}.$$

The right-hand side is exactly $\Lambda^{\mathrm{hl}}_{t,R}(\Phi)$, which vanishes by Corollary 16.4. Hence

$$\Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\Phi_n) \to 0,$$

contradicting $|\Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\Phi_n)| \ge \epsilon_0$. □

Theorem 21.4 (Abstract global reduction from exact fixed-window localization)**.** *Assume the hypotheses of Definition 21.1. Suppose the following localization property holds:*
for every sequence $h_n \downarrow 0$, every sequence

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}) \qquad \textit{with} \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1,$$

and every $\varepsilon > 0$, there exist $R < \infty$, $n_0 \in \mathbb{N}$, and elements

$$\widetilde{\Phi}_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h_n,\alpha,R}) \qquad (n \ge n_0)$$

such that

$$\|\widetilde{\Phi}_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R}} \le 1 + \varepsilon, \tag{252}$$

$$\left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) - \Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\widetilde{\Phi}_n)\right| \le \varepsilon \qquad \textit{for every } n \ge n_0. \tag{253}$$

Then

$$\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} \longrightarrow 0 \qquad \textit{as } h \downarrow 0. \tag{254}$$

Proof. Suppose (254) fails. Then there exist $\epsilon_0 > 0$ and $h_n \downarrow 0$ such that

$$\mathfrak{D}^{\mathrm{mic}}_{t,h_n,\alpha} \ge 4\epsilon_0 \qquad \text{for every } n.$$

By Definition 21.1, choose

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha})$$

with

$$\|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1, \qquad \left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n)\right| \ge 2\epsilon_0.$$

Apply the localization property with $\varepsilon = \epsilon_0$. Then there exist $R < \infty$, n_0, and

$$\widetilde{\Phi}_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{ex}}_{t,h_n,\alpha,R}) \qquad (n \ge n_0)$$

such that

$$\|\widetilde{\Phi}_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R}} \le 1 + \epsilon_0,$$

and
$$\left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) - \Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\widetilde{\Phi}_n)\right| \le \epsilon_0.$$
Hence, for every $n \ge n_0$,
$$\left|\Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\widetilde{\Phi}_n)\right| \ge \epsilon_0.$$
After normalizing by $1+\epsilon_0$, this implies
$$\mathfrak{D}^{\mathrm{ex}}_{t,h_n,\alpha,R} \ge \frac{\epsilon_0}{1+\epsilon_0} \qquad \text{for every } n \ge n_0,$$
which contradicts Theorem 21.3. □

22. Uniform profile tails and buffer selection

The unresolved global step is localization of graph-unit micro sequences on one common finite window without losing the functional. Two preparatory facts follow from the existing theory:

(i) the micro profiles are uniformly tight in $L^2(\mathbb{R}_+)$ as $h \downarrow 0$;
(ii) every graph-unit sequence admits, after subselection, one common annulus on which both the input and output masses are arbitrarily small.

Proposition 22.1 (Uniform tail tightness of the micro profiles)**.** *Assume Assumption 3.1, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and let*
$$i \in \{Y, XY\}.$$
For every $\varepsilon > 0$, there exist $R_\varepsilon < \infty$ and $h_\varepsilon > 0$ such that
$$\left\|\mathbf{1}_{(R_\varepsilon,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon \qquad \textit{for every } 0<h\le h_\varepsilon. \tag{255}$$
Moreover, R_ε and h_ε may be chosen simultaneously for $i = Y$ and $i = XY$.

Proof. Fix $\varepsilon > 0$. By Theorem 14.10,
$$\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} \longrightarrow L_i(t,t)\psi_{H_i} \qquad \text{in } L^2(\mathbb{R}_+)$$
as $h \downarrow 0$. Since $L_i(t,t)\psi_{H_i} \in L^2(\mathbb{R}_+)$, there exists $R_\varepsilon < \infty$ such that
$$\left\|\mathbf{1}_{(R_\varepsilon,\infty)}L_i(t,t)\psi_{H_i}\right\|_{L^2(\mathbb{R}_+)} \le \frac{\varepsilon}{2}.$$
Then choose $h_\varepsilon > 0$ such that
$$\left\|\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} - L_i(t,t)\psi_{H_i}\right\|_{L^2(\mathbb{R}_+)} \le \frac{\varepsilon}{2} \qquad \text{for every } 0<h\le h_\varepsilon.$$
Therefore, for $0 < h \le h_\varepsilon$,
$$\begin{aligned}\left\|\mathbf{1}_{(R_\varepsilon,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha}\right\|_{L^2(\mathbb{R}_+)} &\le \left\|\mathfrak{g}^{\mathrm{mic}}_{i,t,h,\alpha} - L_i(t,t)\psi_{H_i}\right\|_{L^2(\mathbb{R}_+)} \\ &\quad + \left\|\mathbf{1}_{(R_\varepsilon,\infty)}L_i(t,t)\psi_{H_i}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon.\end{aligned}$$
This proves (255). The simultaneous statement for $i = Y$ and $i = XY$ follows by taking the larger of the two radii and the smaller of the two h-thresholds. □

Proposition 22.2 (Annular low-mass buffer selection for graph-unit vectors)**.** *Assume the hypotheses of Definition 21.1, let*
$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi,$$
and suppose
$$\|\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} \le 1.$$
Fix $A \ge 0$, $\ell > 0$, and $M \in \mathbb{N}$, and define the disjoint annuli
$$I_j := [A + (j-1)\ell,\ A + j\ell], \qquad j = 1,\dots,M.$$
Then there exists $j_* \in \{1,\dots,M\}$ such that
$$\|\Phi\|^2_{L^2(I_{j_*})} + \frac{\eta^2}{\nu_Y^2}\|\Psi\|^2_{L^2(I_{j_*})} \le \frac{1}{M}. \tag{256}$$

Proof. Summing over $j = 1, \dots, M$ and using the disjointness of the I_j gives

$$\sum_{j=1}^{M} \left(\|\Phi\|_{L^2(I_j)}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi\|_{L^2(I_j)}^2 \right) \le \|\Phi\|_{L^2(\mathbb{R}_+)}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi\|_{L^2(\mathbb{R}_+)}^2 = \|\Phi\|_{\mathfrak{G}_{t,h,\alpha}^{\mathrm{mic}}}^2 \le 1.$$

Hence at least one summand is at most $1/M$, which is exactly (256). □

Corollary 22.3 (Common low-mass buffers along graph-unit sequences)**.** *Assume the hypotheses of Definition 21.1. Let $h_n \downarrow 0$, and for each n let*

$$\Phi_n \in \mathcal{D}(\mathfrak{T}_{t,h_n,\alpha}^{\mathrm{mic}}), \qquad \Psi_n := \mathfrak{T}_{t,h_n,\alpha}^{\mathrm{mic}} \Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}_{t,h_n,\alpha}^{\mathrm{mic}}} \le 1.$$

Fix $A \ge 0$, $\ell > 0$, and $M \in \mathbb{N}$, and define

$$I_j := [A + (j-1)\ell,\ A + j\ell], \qquad j = 1, \dots, M.$$

Then there exist an index $j_* \in \{1, \dots, M\}$ and a subsequence (n_k) such that

$$\|\Phi_{n_k}\|_{L^2(I_{j_*})}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi_{n_k}\|_{L^2(I_{j_*})}^2 \le \frac{1}{M} \qquad \text{for every } k. \tag{257}$$

Proof. For each n, Proposition 22.2 gives an index

$$j_n \in \{1, \dots, M\}$$

such that

$$\|\Phi_n\|_{L^2(I_{j_n})}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi_n\|_{L^2(I_{j_n})}^2 \le \frac{1}{M}.$$

Since $\{1, \dots, M\}$ is finite, one index j_* occurs infinitely often. Passing to that subsequence gives (257). □

Corollary 22.4 (Profile-tight common buffers)**.** *Assume the hypotheses of Definition 21.1. Let $h_n \downarrow 0$, and let*

$$\Phi_n \in \mathcal{D}(\mathfrak{T}_{t,h_n,\alpha}^{\mathrm{mic}}), \qquad \Psi_n := \mathfrak{T}_{t,h_n,\alpha}^{\mathrm{mic}} \Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}_{t,h_n,\alpha}^{\mathrm{mic}}} \le 1.$$

Fix $\varepsilon > 0$, $\ell > 0$, and $M \in \mathbb{N}$. Then, after passing to a subsequence, there exist finite radii

$$0 < R_0 < R_1 < R_2$$

such that

$$R_1 - R_0 \ge \ell, \qquad R_2 - R_1 = \ell,$$

and:

(i) for $i \in \{Y, XY\}$,

$$\left\| \mathbf{1}_{(R_0,\infty)} \mathfrak{g}_{i,t,h_n,\alpha}^{\mathrm{mic}} \right\|_{L^2(\mathbb{R}_+)} \le \varepsilon \qquad \text{for every } n; \tag{258}$$

(ii) the common transition annulus satisfies

$$\|\Phi_n\|_{L^2([R_1,R_2])}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi_n\|_{L^2([R_1,R_2])}^2 \le \frac{1}{M} \qquad \text{for every } n. \tag{259}$$

Proof. By Proposition 22.1, after discarding finitely many indices there exists $R_0 < \infty$ such that (258) holds for both $i = Y$ and $i = XY$ for every remaining n. Set

$$A := R_0 + \ell.$$

Applying Corollary 22.3 with this A, the same ℓ, and the same M, and then relabeling the resulting subsequence, yields an index $j_* \in \{1, \dots, M\}$ such that

$$\|\Phi_n\|_{L^2(I_{j_*})}^2 + \frac{\eta^2}{\nu_Y^2} \|\Psi_n\|_{L^2(I_{j_*})}^2 \le \frac{1}{M} \qquad \text{for every } n,$$

where

$$I_{j_*} = [A + (j_* - 1)\ell,\ A + j_*\ell].$$

Now define
$$R_1 := A + (j_* - 1)\ell, \qquad R_2 := A + j_*\ell.$$
Then $R_2 - R_1 = \ell$, $R_1 \geq A = R_0 + \ell$, and therefore $R_1 - R_0 \geq \ell$. The annulus estimate (259) is exactly the displayed bound above. □

23. Non-escaping sequences and functional concentration

Common low-mass buffers do not by themselves localize every sequence. If a graph-unit micro sequence escapes to infinity, the micro functional vanishes because the profiles are uniformly tight. The remaining regime is the non-escaping one, where a fixed amount of graph mass stays on some common finite window.

Proposition 23.1 (Escape to infinity forces functional collapse)**.** *Assume the hypotheses of Definition 21.1. Let $h_n \downarrow 0$, and let*
$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \leq 1.$$
Assume moreover that for every $R < \infty$,
$$\|\Phi_n\|_{L^2([0,R])} + \|\Psi_n\|_{L^2([0,R])} \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{260}$$
Then
$$\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) \longrightarrow 0 \qquad \text{as } n \to \infty. \tag{261}$$

Proof. Fix $\varepsilon > 0$. By Proposition 22.1, there exist R_ε and n_ε such that for every $n \geq n_\varepsilon$,
$$\left\|\mathbf{1}_{(R_\varepsilon,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h_n,\alpha}\right\|_{L^2(\mathbb{R}_+)} \leq \varepsilon \qquad \text{for } i \in \{Y, XY\}. \tag{262}$$
By (260), after increasing n_ε if necessary, we may also assume that for every $n \geq n_\varepsilon$,
$$\|\Phi_n\|_{L^2([0,R_\varepsilon])} + \|\Psi_n\|_{L^2([0,R_\varepsilon])} \leq \varepsilon. \tag{263}$$
Using (245), split both pairings at R_ε. For $n \geq n_\varepsilon$,
$$\begin{aligned}\left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n)\right| \leq{}& \left\|\mathbf{1}_{[0,R_\varepsilon]}\mathfrak{g}^{\mathrm{mic}}_{XY,t,h_n,\alpha}\right\|_{L^2} \|\Phi_n\|_{L^2([0,R_\varepsilon])} \\ &+ \left\|\mathbf{1}_{(R_\varepsilon,\infty)}\mathfrak{g}^{\mathrm{mic}}_{XY,t,h_n,\alpha}\right\|_{L^2} \|\Phi_n\|_{L^2(\mathbb{R}_+)} \\ &+ \left\|\mathbf{1}_{[0,R_\varepsilon]}\mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}\right\|_{L^2} \|\Psi_n\|_{L^2([0,R_\varepsilon])} \\ &+ \left\|\mathbf{1}_{(R_\varepsilon,\infty)}\mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}\right\|_{L^2} \|\Psi_n\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$
The full L^2 norms of the micro profiles are uniformly bounded for large n by Theorem 14.10. Also,
$$\|\Phi_n\|_{L^2(\mathbb{R}_+)} \leq 1, \qquad \|\Psi_n\|_{L^2(\mathbb{R}_+)} \leq \frac{\nu_Y}{\eta},$$
because $\|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \leq 1$. Combining these bounds with (262) and (263) yields
$$\limsup_{n\to\infty} \left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n)\right| \leq C_{t,\eta,\nu_Y}\varepsilon.$$
Since $\varepsilon > 0$ was arbitrary, (261) follows. □

Corollary 23.2 (Non-escaping alternative for nonvanishing functional sequences)**.** *Assume the hypotheses of Definition 21.1. Let $h_n \downarrow 0$, and let*
$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \leq 1.$$
Suppose that
$$\limsup_{n\to\infty} \left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n)\right| > 0. \tag{264}$$
Then there exist $R_* < \infty$, $\delta_* > 0$, and a subsequence, not relabeled, such that
$$\|\Phi_n\|_{L^2([0,R_*])} + \|\Psi_n\|_{L^2([0,R_*])} \geq \delta_* \qquad \text{for every } n. \tag{265}$$

Proof. If no such R_* and δ_* existed, then after passing to a subsequence one would have (260) for every $R < \infty$. Proposition 23.1 would then imply $\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) \to 0$, contradicting (264). $\square$

Corollary 23.3 (Structured reduction of the global localization problem)**.** *Assume the hypotheses of Definition 21.1. To verify the localization hypothesis in Theorem 21.4, it is enough to treat graph-unit sequences*

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1,$$

for which, after passage to a subsequence, there exist:

(i) a finite radius $R_* < \infty$ and a constant $\delta_* > 0$ such that

$$\|\Phi_n\|_{L^2([0,R_*])} + \|\Psi_n\|_{L^2([0,R_*])} \ge \delta_* \qquad \text{for every } n;$$

(ii) for every prescribed $\varepsilon > 0$, $\ell > 0$, and $M \in \mathbb{N}$, finite radii

$$0 < R_0 < R_1 < R_2$$

such that

$$R_1 - R_0 \ge \ell, \qquad R_2 - R_1 = \ell,$$

and the profile-tail and good-buffer bounds (258) *and* (259) *hold.*

Proof. If $\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) \to 0$ already, then the localization hypothesis in Theorem 21.4 is already satisfied: for any $\varepsilon > 0$, taking $\widetilde{\Phi}_n := 0$ on any fixed window gives

$$\|\widetilde{\Phi}_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R}} = 0 \le 1 + \varepsilon$$

and, for large n,

$$\left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) - \Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R}(\widetilde{\Phi}_n)\right| = \left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n)\right| \le \varepsilon.$$

Thus only sequences satisfying (264) need to be considered. For such sequences, Corollary 23.2 gives the non-escaping head bound, and Corollary 22.4 then gives the common profile-tight buffers after passing to a further subsequence. $\square$

24. Localization from full-window defect control

The final global reduction has a definite form. A smoothly truncated global exact micro pair must have a small *full-window* conjugated defect on one common finite window. The correction theorem and the profile-tightness estimates then upgrade the truncated pair to an exact fixed-window approximant with small norm loss and small functional loss.

Theorem 24.1 (Exact fixed-window localization from a small full-window defect)**.** *Assume the hypotheses of Definition 21.1. Let $h_n \downarrow 0$, and let*

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1.$$

Fix radii

$$0 < R_0 < R_1 < R_2 < \infty,$$

and let χ be a smooth buffered cutoff with radii (R_0, R_1, R_2). Assume that for some $\varepsilon > 0$ and all sufficiently large n,

$$\left\|\mathbf{1}_{(R_1,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h_n,\alpha}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon \qquad \text{for } i \in \{Y, XY\}, \tag{266}$$

$$\left\|\mathfrak{F}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}(\chi\Phi_n, \chi\Psi_n)\right\|_{E_{R_2}} \le \varepsilon. \tag{267}$$

Define

$$\widetilde{\Psi}_n := \chi\Psi_n - \left(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h_n,\alpha,R_2}\right)^{-1}\mathfrak{F}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}(\chi\Phi_n, \chi\Psi_n), \tag{268}$$

and set

$$\widetilde{\Phi}_n := \chi\Phi_n.$$

Then there exists $C_{t,R_2,\eta,\nu_Y} < \infty$ such that, for all sufficiently large n,

(i)

$$(\widetilde{\Phi}_n, \widetilde{\Psi}_n) \in \Gamma^{\mathrm{ex}}_{t,h_n,\alpha,R_2};$$

(ii)

$$\|\widetilde{\Phi}_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}} \le 1 + C_{t,R_2,\eta,\nu_Y}\varepsilon; \tag{269}$$

(iii)

$$\left|\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) - \Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R_2}(\widetilde{\Phi}_n)\right| \le C_{t,R_2,\eta,\nu_Y}\varepsilon. \tag{270}$$

Proof. Part (i) is exactly Theorem 19.6 applied to the pair $(\chi\Phi_n, \chi\Psi_n)$.

For part (ii), $|\chi| \le 1$ gives

$$\|\widetilde{\Phi}_n\|_{E_{R_2}} \le \|\Phi_n\|_{L^2(\mathbb{R}_+)} \le 1.$$

By Theorem 19.6 and (267),

$$\|\widetilde{\Psi}_n - \chi\Psi_n\|_{E_{R_2}} \le C_{t,R_2}\varepsilon.$$

Since $|\chi| \le 1$ and $\|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1$,

$$\|\chi\Psi_n\|_{E_{R_2}} \le \|\Psi_n\|_{L^2(\mathbb{R}_+)} \le \frac{\nu_Y}{\eta}.$$

Therefore using $\|\Phi_n\|^2_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1$,

$$\begin{aligned}
\|\widetilde{\Phi}_n\|^2_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}} &= \|\widetilde{\Phi}_n\|^2_{E_{R_2}} + \frac{\eta^2}{\nu_Y^2}\|\widetilde{\Psi}_n\|^2_{E_{R_2}} \\
&\le \|\Phi_n\|^2_{L^2(\mathbb{R}_+)} + \frac{\eta^2}{\nu_Y^2}\big(\|\Psi_n\|_{L^2(\mathbb{R}_+)} + C_{t,R_2}\varepsilon\big)^2 \\
&\le \|\Phi_n\|^2_{L^2(\mathbb{R}_+)} + \frac{\eta^2}{\nu_Y^2}\|\Psi_n\|^2_{L^2(\mathbb{R}_+)} + C_{t,R_2,\eta,\nu_Y}\varepsilon \\
&\le 1 + C_{t,R_2,\eta,\nu_Y}\varepsilon.
\end{aligned}$$

After enlarging the constant if necessary,

$$\|\widetilde{\Phi}_n\|_{\mathfrak{G}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}} \le 1 + C_{t,R_2,\eta,\nu_Y}\varepsilon,$$

which proves (269).

For part (iii), using (245), (246), and part (i), one has

$$\begin{aligned}
\Lambda^{\mathrm{mic}}_{t,h_n,\alpha}(\Phi_n) - \Lambda^{\mathrm{ex}}_{t,h_n,\alpha,R_2}(\widetilde{\Phi}_n) &= \Big\langle \mathfrak{g}^{\mathrm{mic}}_{XY,t,h_n,\alpha}, (1-\chi)\Phi_n \Big\rangle_{L^2(\mathbb{R}_+)} \\
&\quad - \Big\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}, (1-\chi)\Psi_n \Big\rangle_{L^2(\mathbb{R}_+)} \\
&\quad - \Big\langle \mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}, \widetilde{\Psi}_n - \chi\Psi_n \Big\rangle_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

Because $(1-\chi)$ is supported in $[R_1, \infty)$, the first two terms are bounded by

$$\left\|\mathbf{1}_{(R_1,\infty)}\mathfrak{g}^{\mathrm{mic}}_{XY,t,h_n,\alpha}\right\|_{L^2} \|\Phi_n\|_{L^2(\mathbb{R}_+)} + \left\|\mathbf{1}_{(R_1,\infty)}\mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}\right\|_{L^2} \|\Psi_n\|_{L^2(\mathbb{R}_+)}.$$

Using (266) and

$$\|\Phi_n\|_{L^2(\mathbb{R}_+)} \le 1, \qquad \|\Psi_n\|_{L^2(\mathbb{R}_+)} \le \frac{\nu_Y}{\eta},$$

these terms are at most $C_{\eta,\nu_Y}\varepsilon$. For the correction term, the full L^2-norm of $\mathfrak{g}^{\mathrm{mic}}_{Y,t,h_n,\alpha}$ is uniformly bounded for large n by Theorem 14.10, while Theorem 19.6 and (267) give

$$\|\widetilde{\Psi}_n - \chi\Psi_n\|_{E_{R_2}} \le C_{t,R_2}\varepsilon.$$

Hence the third term is also bounded by $C_{t,R_2,\eta,\nu_Y}\varepsilon$, which proves (270). □

Corollary 24.2 (Reduction of abstract localization to full-window defect control)**.** *Assume the hypotheses of Definition 21.1. To verify the localization hypothesis in Theorem 21.4, it is enough to prove the following statement:*

for every sequence $h_n \downarrow 0$ and every graph-unit sequence

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1,$$

which satisfies the structured reduction of Corollary 23.3, and for every $\varepsilon > 0$, there exist common radii

$$0 < R_0 < R_1 < R_2 < \infty$$

and one smooth buffered cutoff χ with radii (R_0, R_1, R_2) such that, after passing to a subsequence if needed, (266) *and* (267) *hold.*

Proof. By Corollary 23.3, it is enough to treat non-escaping sequences with a common head window and common profile-tight buffers. For such sequences, if (266) and (267) hold, then Theorem 24.1 produces exact fixed-window approximants on the common window $[0, R_2]$ with arbitrarily small losses in both graph norm and functional value. This gives the localization property required in Theorem 21.4. □

25. Annulus-outer splitting of the full-window defect

The full-window defect has two sources: the transition annulus and the outer tail. Because the cutoff is constant on $[0, R_1]$, mass below R_1 does not enter the commutator. The annulus $[R_1, R_2]$ is therefore the only local contribution, and its size is reduced to an explicit quantitative bound. The complementary contribution is the outer tail (R_2, ∞).

Proposition 25.1 (Annulus-outer decomposition of the full-window defect)**.** *Assume the hypotheses of Definition 21.1. Let*

$$0 < R_0 < R_1 < R_2 < \infty,$$

let χ be a smooth buffered cutoff with radii (R_0, R_1, R_2), and let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Assume in addition that

$$hR_2 < t, \qquad R_2 < \varepsilon_{h,\alpha}^{-1}.$$

Define

$$\Phi^{\mathrm{ann}} := \Phi\mathbf{1}_{[R_1,R_2]}, \qquad \Phi^{\mathrm{out}} := \Phi\mathbf{1}_{(R_2,\infty)}, \qquad \Psi^{\mathrm{ann}} := \Psi\mathbf{1}_{[R_1,R_2]}, \qquad \Psi^{\mathrm{out}} := \Psi\mathbf{1}_{(R_2,\infty)}.$$

Then

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi, \chi\Psi) = \mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi) + \mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi) \qquad \text{in } E_{R_2}, \tag{271}$$

where

$$\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi) := \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{ann}} - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{ann}}, \tag{272}$$

$$\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi) := \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{out}} - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{out}}. \tag{273}$$

Moreover, there exists $C_{t,R_2} < \infty$ such that

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi)\right\|_{E_{R_2}} \le C_{t,R_2}\Lambda_{\chi,R_2}\big(\|\Phi\|_{L^2([R_1,R_2])} + \|\Psi\|_{L^2([R_1,R_2])}\big), \tag{274}$$

where

$$\Lambda_{\chi,R_2} := \left\|\frac{\chi(q)-\chi(r)}{q-r}\right\|_{C^1(\Delta_{R_2})}, \qquad \Delta_{R_2} := \{0 \le r \le q \le R_2\}.$$

If $\chi = \chi_{R_1,R_2}$ is the canonical cutoff from Lemma 20.1, then

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi_{R_1,R_2}}(\Phi,\Psi)\right\|_{E_{R_2}} \le C_{t,R_2}\left(\frac{1}{R_2-R_1} + \frac{1}{(R_2-R_1)^2}\right)\big(\|\Phi\|_{L^2([R_1,R_2])} + \|\Psi\|_{L^2([R_1,R_2])}\big). \tag{275}$$

Proof. By Theorem 19.9(i),

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi) = \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi \qquad \text{in } E_{R_2}.$$

Since

$$\Phi = \Phi\mathbf{1}_{[0,R_1]} + \Phi^{\mathrm{ann}} + \Phi^{\mathrm{out}}, \qquad \Psi = \Psi\mathbf{1}_{[0,R_1]} + \Psi^{\mathrm{ann}} + \Psi^{\mathrm{out}},$$

linearity gives

$$\begin{aligned}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_2}(\chi\Phi,\chi\Psi) &= \Big(\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}(\Psi\mathbf{1}_{[0,R_1]}) - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}(\Phi\mathbf{1}_{[0,R_1]})\Big)\\ &\quad + \mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi) + \mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi).\end{aligned}$$

It therefore remains to show that the first bracket vanishes.

Let $\Theta \in L^2(\mathbb{R}_+)$ be supported in $[0,R_1]$. If the integrand in (228) or (229) is nonzero, then

$$r \le q \le R_1.$$

Because $\chi \equiv 1$ on $[0,R_1]$, one has $\chi(q) - \chi(r) = 0$. Therefore

$$\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta = \mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta = 0 \qquad \text{on } [0,R_2].$$

Applying (230) and (231) gives

$$\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta = \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta = 0 \qquad \text{in } E_{R_2},$$

which proves (271).

For (274), note that $\Phi^{\mathrm{ann}}, \Psi^{\mathrm{ann}} \in E_{R_2}$, so Proposition 20.2 applies:

$$\begin{aligned}\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi)\right\|_{E_{R_2}} &\le \left\|\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{ann}}\right\|_{E_{R_2}} + \left\|\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{ann}}\right\|_{E_{R_2}}\\ &\le C_{t,R_2}\Lambda_{\chi,R_2}\big(\|\Psi^{\mathrm{ann}}\|_{E_{R_2}} + \|\Phi^{\mathrm{ann}}\|_{E_{R_2}}\big),\end{aligned}$$

which is exactly (274). The width estimate (275) follows from (274) and Lemma 20.1. $\square$

Proposition 25.2 (Uniform control of the annulus commutator)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Fix $\ell > 0$. Then there exists $C_{t,\ell} < \infty$ such that the following holds. Let

$$0 < R_1 < R_2 < \infty, \qquad R_2 - R_1 = \ell,$$

let $\chi := \chi_{R_1,R_2}$ be the canonical cutoff from Lemma 20.1, and choose $h > 0$ sufficiently small that

$$hR_2 < t, \qquad R_2 < \varepsilon_{h,\alpha}^{-1}.$$

Then for every $\Theta \in L^2([R_1,R_2])$,

$$\left\|\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} + \left\|\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}. \tag{276}$$

Consequently, for every

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi,$$

one has

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}(\Phi,\Psi)\right\|_{E_{R_2}} \le C_{t,\ell}\big(\|\Phi\|_{L^2([R_1,R_2])} + \|\Psi\|_{L^2([R_1,R_2])}\big). \tag{277}$$

Proof. Let

$$S_0 := (R_1 - \ell)_+, \qquad J := [S_0, R_2].$$

Then $|J| \le 2\ell$, and the support of every $\Theta \in L^2([R_1,R_2])$ lies inside J.

We first control the output on the local strip J. Since Θ is supported in $[R_1,R_2]$ and $\chi = \chi_{R_1,R_2}$ satisfies the derivative bounds (236), the proof of Proposition 20.2, restricted to the strip J, gives

$$\left\|\mathbf{1}_J\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} + \left\|\mathbf{1}_J\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}, \tag{278}$$

because the output interval has length at most 2ℓ, the input interval has length ℓ, and the singular integrals in the source-side estimate are therefore taken over a set whose size is controlled solely by ℓ.

It remains to control the far-left part $[0, S_0]$. If $S_0 = 0$, there is nothing to prove, so assume $S_0 > 0$, equivalently $R_1 > \ell$. For $0 \le r \le S_0$ and $q \in [R_1, R_2]$, one has $\chi(r) = 1$, hence

$$\chi(q) - \chi(r) = -(1 - \chi(q)).$$

Therefore, for every $\Theta \in L^2([R_1, R_2])$, defining

$$\widetilde{\Theta} := (1 - \chi)\Theta \in L^2([R_1, R_2]),$$

the commutator Volterra terms reduce on $[0, S_0]$ to

$$\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta = -\mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,S_0,R_1}\widetilde{\Theta},$$
$$\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta = -\mathfrak{W}^{\mathrm{tail}}_{XY,t,h,\alpha,S_0,R_1}\widetilde{\Theta}.$$

The proofs of Propositions 17.2 and 17.3 now apply with the additional information that $\widetilde{\Theta}$ is supported in the interval $[R_1, R_2]$ of length ℓ. On the target side, the conjugated kernel is uniformly $O(h)$, so

$$\left\|\mathbf{1}_{[0,S_0]}\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_t h\sqrt{S_0\ell}\,\|\Theta\|_{L^2([R_1,R_2])}.$$

Since $S_0 \le R_2 < \varepsilon_{h,\alpha}^{-1} = h^{\alpha-1}$, this gives

$$\left\|\mathbf{1}_{[0,S_0]}\mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}. \tag{279}$$

On the source side, the proof of Proposition 17.3 yields a decomposition of the conjugated kernel into a main term bounded by $C_t(q-r)^{-\beta-1}$ and an error bounded by $C_t h(q-r)^{-\beta}$. Since $r \le S_0 = R_1 - \ell$ and $q \in [R_1, R_2]$, one has

$$q - r \ge \ell.$$

Schur's test therefore bounds the contribution of the main kernel by

$$C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}.$$

For the error kernel, the Hilbert–Schmidt estimate gives

$$C_t h\sqrt{S_0}\,\ell^{\frac{1}{2}-\beta}\|\Theta\|_{L^2([R_1,R_2])} \le C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}.$$

Hence

$$\left\|\mathbf{1}_{[0,S_0]}\mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Theta\right\|_{E_{R_2}} \le C_{t,\ell}\|\Theta\|_{L^2([R_1,R_2])}. \tag{280}$$

Combining (278), (279), and (280) proves (276). The pair estimate (277) follows from the definition (272) and the triangle inequality. □

Corollary 25.3 (Good buffers imply annulus control)**.** *Assume the hypotheses of Definition 21.1. For every $\varepsilon > 0$ and every $\ell > 0$, after passing to a subsequence, there exist common radii*

$$0 < R_0 < R_1 < R_2 < \infty, \qquad R_2 - R_1 = \ell,$$

such that:

(i) for $i \in \{Y, XY\}$,

$$\left\|\mathbf{1}_{(R_1,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h_n,\alpha}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon \qquad \textit{for every } n;$$

(ii) for the canonical cutoff χ_{R_1,R_2},

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} \le \varepsilon \qquad \textit{for every } n. \tag{281}$$

Proof. Let $C_{t,\ell}$ be the constant from Proposition 25.2, and choose $M \in \mathbb{N}$ so large that

$$C_{t,\ell}\left(1+\frac{\nu_Y}{\eta}\right)M^{-1/2} \le \varepsilon.$$

Applying Corollary 22.4 with this M, with this ℓ, and with the same ε, and then relabeling the resulting subsequence, yields common radii

$$0 < R_0 < R_1 < R_2 < \infty, \qquad R_1 - R_0 \ge \ell, \qquad R_2 - R_1 = \ell,$$

such that the profile-tail estimate beyond R_0 holds and

$$\|\Phi_n\|^2_{L^2([R_1,R_2])} + \frac{\eta^2}{\nu_Y^2}\|\Psi_n\|^2_{L^2([R_1,R_2])} \le \frac{1}{M} \qquad \text{for every } n.$$

Since $R_1 > R_0$, the same profile-tail bound holds beyond R_1, which proves (i). The annulus mass bound implies

$$\|\Phi_n\|_{L^2([R_1,R_2])} + \|\Psi_n\|_{L^2([R_1,R_2])} \le \left(1+\frac{\nu_Y}{\eta}\right)M^{-1/2},$$

and after discarding finitely many terms we may also assume

$$h_n R_2 < t, \qquad R_2 < \varepsilon^{-1}_{h_n,\alpha} \qquad \text{for every } n.$$

Proposition 25.2 then gives (281). $\square$

Lemma 25.4 (Uniform separated buffered tail bounds)**.** *Assume the hypotheses of Definition 17.1. Then there exist constants*

$$C_t < \infty, \qquad C_{t,\alpha} < \infty$$

such that for every sufficiently small $h > 0$,

$$\left\|\mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_0,R}\right\|_{\mathcal{L}(F_{h,R},E_{R_0})} \le C_t h^\alpha, \tag{282}$$

$$\left\|\mathfrak{Q}^{\mathrm{tail}}_{XY,t,h,\alpha,R_0,R}\right\|_{\mathcal{L}(F_{h,R},E_{R_0})} \le C_t(R-R_0)^{-\beta} + C_{t,\alpha}h^{\alpha+\beta-\alpha\beta}. \tag{283}$$

Proof. The target-side estimate follows from (203) and the proof of Proposition 17.2: since $R_0 < R < \varepsilon^{-1}_{h,\alpha}$,

$$h\sqrt{\varepsilon^{-1}_{h,\alpha} - R} \le h\sqrt{\varepsilon^{-1}_{h,\alpha}} = h^{\frac{1+\alpha}{2}} \le h^\alpha \qquad (0 < h \le 1),$$

which proves (282).

For the source side, the main term is exactly (206), hence bounded by $C_t(R-R_0)^{-\beta}$. For the error term, the proof of Proposition 17.3 gives

$$\left\|K^{(2)}_{XY,t,h}\right\|^2_{L^2([0,R_0]\times[R,\varepsilon^{-1}_{h,\alpha}])} \le C_t h^2 R_0\, \varepsilon^{-(1-2\beta)}_{h,\alpha}.$$

Since $R_0 < \varepsilon^{-1}_{h,\alpha}$, one obtains

$$\left\|K^{(2)}_{XY,t,h}\right\|_{\mathcal{L}(F_{h,R},E_{R_0})} \le C_t h\, \varepsilon^{-(1-\beta)}_{h,\alpha} = C_{t,\alpha}h^{\alpha+\beta-\alpha\beta},$$

which proves (283). $\square$

Proposition 25.5 (Separated part of the outer-tail commutator)**.** *Assume the hypotheses of Definition 21.1. Let*

$$0 < R_1 < R_2 < \infty, \qquad \ell := R_2 - R_1,$$

let $\chi := \chi_{R_1,R_2}$ be the canonical cutoff from Lemma 20.1, and let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Assume

$$hR_2 < t, \qquad R_2 < \varepsilon^{-1}_{h,\alpha}.$$

Then

$$\mathbf{1}_{[0,R_1]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi) = \mathfrak{Q}^{\mathrm{tail}}_{XY,t,h,\alpha,R_1,R_2}\Phi^{\mathrm{out}} - \mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_1,R_2}\Psi^{\mathrm{out}} \qquad \textit{in } E_{R_1}, \tag{284}$$

and there exist constants $C_t < \infty$ and $C_{t,\alpha,\eta,\nu_Y} < \infty$ such that

$$\left\|\mathbf{1}_{[0,R_1]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi)\right\|_{E_{R_2}} \le C_t\ell^{-\beta} + C_{t,\alpha,\eta,\nu_Y}h^{\alpha} \tag{285}$$

for all sufficiently small $h > 0$.

Proof. For $0 \le r \le R_1$ and $q \ge R_2$, one has $\chi(r) = 1$ and $\chi(q) = 0$. Hence

$$\chi(q) - \chi(r) = -1.$$

Therefore, on $[0, R_1]$,

$$\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{out}} = -\mathfrak{W}^{\mathrm{tail}}_{Y,t,h,\alpha,R_1,R_2}\Psi^{\mathrm{out}},$$

and similarly

$$\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{out}} = -\mathfrak{W}^{\mathrm{tail}}_{XY,t,h,\alpha,R_1,R_2}\Phi^{\mathrm{out}}.$$

Applying (230), (231), (202), and (204) yields (284).

Now use Lemma 25.4 with $R_0 = R_1$ and $R = R_2$:

$$\begin{aligned}\left\|\mathbf{1}_{[0,R_1]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi)\right\|_{E_{R_2}} &\le \left\|\mathfrak{Q}^{\mathrm{tail}}_{XY,t,h,\alpha,R_1,R_2}\Phi^{\mathrm{out}}\right\|_{E_{R_1}} + \left\|\mathfrak{Q}^{\mathrm{tail}}_{Y,t,h,\alpha,R_1,R_2}\Psi^{\mathrm{out}}\right\|_{E_{R_1}} \\ &\le (C_t\ell^{-\beta} + C_{t,\alpha}h^{\alpha+\beta-\alpha\beta})\|\Phi^{\mathrm{out}}\|_{L^2(\mathbb{R}_+)} + C_t h^{\alpha}\|\Psi^{\mathrm{out}}\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Since

$$\|\Phi^{\mathrm{out}}\|_{L^2(\mathbb{R}_+)} \le \|\Phi\|_{L^2(\mathbb{R}_+)} \le 1, \qquad \|\Psi^{\mathrm{out}}\|_{L^2(\mathbb{R}_+)} \le \|\Psi\|_{L^2(\mathbb{R}_+)} \le \frac{\nu_Y}{\eta},$$

and $h^{\alpha+\beta-\alpha\beta} \le h^{\alpha}$ for $0 < h \le 1$, this proves (285). □

Corollary 25.6 (Localization reduces to boundary-strip outer-tail control)**.** *Assume the hypotheses of Definition 21.1. To verify the localization hypothesis in Theorem 21.4, it is enough to prove the following statement:*

for every sequence $h_n \downarrow 0$, every graph-unit sequence

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1,$$

which satisfies the structured reduction of Corollary 23.3, and every $\varepsilon > 0$, there exists $\ell > 0$ such that after passing to a subsequence and fixing the common radii

$$0 < R_0 < R_1 < R_2 < \infty, \qquad R_2 - R_1 = \ell,$$

and the canonical cutoff $\chi = \chi_{R_1,R_2}$ supplied by Corollary 25.3, one has

$$\left\|\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi}(\Phi_n,\Psi_n)\right\|_{E_{R_2}} \le \varepsilon \qquad \textit{for every } n. \tag{286}$$

Proof. Choose $\ell > 0$ so large that

$$C_t\ell^{-\beta} \le \varepsilon,$$

where C_t is the constant from Proposition 25.5. By Corollary 25.3, after passing to a subsequence we may assume that the common profile-tail bound holds beyond R_1 and that

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h_n,\alpha,\chi}(\Phi_n,\Psi_n)\right\|_{E_{R_2}} \le \varepsilon \qquad \text{for every } n.$$

After discarding finitely many terms one also has

$$h_nR_2 < t, \qquad R_2 < \varepsilon^{-1}_{h_n,\alpha} \qquad \text{for every } n,$$

and

$$C_{t,\alpha,\eta,\nu_Y}h_n^{\alpha} \le \varepsilon \qquad \text{for every } n.$$

Hence Proposition 25.5 gives

$$\left\|\mathbf{1}_{[0,R_1]}\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi}(\Phi_n,\Psi_n)\right\|_{E_{R_2}} \le 2\varepsilon \qquad \text{for every } n.$$

If (286) also holds, then

$$\left\|\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi}(\Phi_n,\Psi_n)\right\|_{E_{R_2}} \le 3\varepsilon \qquad \text{for every } n.$$

Together with the annulus bound and Proposition 25.1, this gives

$$\left\|\mathfrak{F}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}(\chi\Phi_n,\chi\Psi_n)\right\|_{E_{R_2}} \le 4\varepsilon \qquad \text{for every } n.$$

The profile-tail bound beyond R_1 and Theorem 24.1 therefore give exact fixed-window approximants with arbitrarily small losses in norm and functional value. By Corollary 24.2, this verifies the localization hypothesis in Theorem 21.4. □

Remark 25.7 (Explicit form of the annulus term). Proposition 25.1 removes the annulus contribution from the list of hidden unknowns. It is the explicit operator $\mathfrak{A}^{\mathrm{ann}}_{t,h,\alpha,\chi}$, and Proposition 25.2 and Corollary 25.3 now show that this annulus term can be made uniformly small along structured non-escaping sequences. Thus the remaining global work is reduced to the strip contribution $\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}$. As Section 28 shows, this is no longer a tail-driven quantity once the exact micro transfer identity is used.

26. Flat outer-edge structure of the canonical cutoff

The boundary-strip term is created where the canonical cutoff meets the outer edge R_2. The needed property is stronger than C^2. Because the profile used in Lemma 20.1 is identically zero on $[1,\infty)$, the rescaled cutoff is flat to every order at R_2. This flatness softens the conjugated commutator kernels in the strip analysis.

Lemma 26.1 (Arbitrary-order flatness at the outer edge)**.** *For every pair of integers $m \ge 0$ and $k \in \{0,1,2\}$, there exists a constant*

$$C_{\chi,m,k} < \infty$$

such that for every $0 < R_1 < R_2 < \infty$, the canonical cutoff χ_{R_1,R_2} from Lemma 20.1 satisfies

$$\left|\chi^{(k)}_{R_1,R_2}(r)\right| \le C_{\chi,m,k}\frac{(R_2-r)^m}{(R_2-R_1)^{m+k}} \qquad \textit{for every } r \in [R_1,R_2]. \tag{287}$$

Proof. Recall from the proof of Lemma 20.1 that

$$\chi_{R_1,R_2}(r) = \vartheta\left(\frac{r-R_1}{R_2-R_1}\right), \qquad r \ge 0,$$

for some fixed $\vartheta \in C^\infty(\mathbb{R})$ such that

$$\vartheta \equiv 1 \text{ on } (-\infty,0], \qquad \vartheta \equiv 0 \text{ on } [1,\infty).$$

Hence, for every $k \in \{0,1,2\}$, the derivative $\vartheta^{(k)}$ vanishes identically on $[1,\infty)$, and therefore all one-sided derivatives of $\vartheta^{(k)}$ at 1 vanish. Since $\vartheta^{(k)}$ is smooth on a neighborhood of 1, Taylor's theorem at the point 1 shows that for every $m \ge 0$ there exists $C_{\chi,m,k} < \infty$ such that

$$\left|\vartheta^{(k)}(u)\right| \le C_{\chi,m,k}(1-u)^m \qquad \text{for every } u \in [0,1].$$

Now let

$$u := \frac{r-R_1}{R_2-R_1} \in [0,1].$$

Then

$$\chi^{(k)}_{R_1,R_2}(r) = \frac{1}{(R_2-R_1)^k}\vartheta^{(k)}(u),$$

and

$$1-u = \frac{R_2-r}{R_2-R_1}.$$

Substituting these identities into the previous bound yields (287). □

Corollary 26.2 (Boundary-strip decay of the cutoff quotient)**.** *Let $0 < R_1 < R_2 < \infty$, set $\ell := R_2 - R_1$, and let χ_{R_1,R_2} be the canonical cutoff from Lemma 20.1. For $r \in [R_1,R_2]$ and $q \ge R_2$, define*

$$\delta_{R_1,R_2}(r,q) := \frac{\chi_{R_1,R_2}(q)-\chi_{R_1,R_2}(r)}{q-r} = -\frac{\chi_{R_1,R_2}(r)}{q-r}.$$

Then, for every integer $m \geq 0$, there exists $C_{\chi,m} < \infty$ such that

$$|\delta_{R_1,R_2}(r,q)| \leq C_{\chi,m} \frac{(R_2 - r)^m}{\ell^m (q-r)}, \tag{288}$$

$$|\partial_r \delta_{R_1,R_2}(r,q)| + |\partial_q \delta_{R_1,R_2}(r,q)| \leq C_{\chi,m} \left(\frac{(R_2-r)^m}{\ell^{m+1}(q-r)} + \frac{(R_2-r)^m}{\ell^m (q-r)^2} \right). \tag{289}$$

Proof. The identity

$$\delta_{R_1,R_2}(r,q) = -\frac{\chi_{R_1,R_2}(r)}{q-r}$$

follows because $\chi_{R_1,R_2}(q) = 0$ for $q \geq R_2$. The bound (288) then follows from (287) with $k = 0$.

For the derivatives, direct differentiation gives

$$\partial_r \delta_{R_1,R_2}(r,q) = -\frac{\chi'_{R_1,R_2}(r)}{q-r} - \frac{\chi_{R_1,R_2}(r)}{(q-r)^2}, \qquad \partial_q \delta_{R_1,R_2}(r,q) = \frac{\chi_{R_1,R_2}(r)}{(q-r)^2}.$$

Applying (287) with $k = 0$ and $k = 1$ yields (289). □

27. Crossing-segment geometry at the outer edge

The remaining boundary-strip term comes from Volterra segments that start at $r \in [R_1, R_2]$ in the transition annulus and end at $q \geq R_2$. The canonical cutoff vanishes once the segment crosses R_2, so each weighted Beta-average samples only the initial fraction

$$\rho_{R_1,R_2}(r,q) := \frac{R_2 - r}{q - r} \in [0,1].$$

The strip analysis combines this truncation geometry with the exact conjugated Volterra formulas.

Lemma 27.1 (Weighted segment truncation across the outer edge)**.** *Let $a < 1$, $b > -1$, $m \in \mathbb{N}_0$, and $k \in \{0,1\}$. Then there exists a constant*

$$C_{a,b,m,k} < \infty$$

such that for every $0 < R_1 < R_2 < \infty$, with $\ell := R_2 - R_1$, the canonical cutoff χ_{R_1,R_2} from Lemma 20.1 satisfies

$$\int_0^1 \theta^{-a}(1-\theta)^b \left| \chi^{(k)}_{R_1,R_2}(r + (q-r)\theta) \right| d\theta \leq C_{a,b,m,k}\, \ell^{-m-k} (R_2 - r)^m \, \rho_{R_1,R_2}(r,q)^{1-a} \tag{290}$$

for every $r \in [R_1, R_2]$ and every $q \geq R_2$.

Proof. Fix $r \in [R_1, R_2]$ and $q \geq R_2$, and set $\rho := \rho_{R_1,R_2}(r,q)$. Then

$$r + (q-r)\theta \geq R_2 \qquad \text{whenever } \theta \geq \rho.$$

Because χ_{R_1,R_2} vanishes identically on $[R_2, \infty)$, the integrand in (290) is zero for $\theta \geq \rho$. Hence

$$\int_0^1 \theta^{-a}(1-\theta)^b \left| \chi^{(k)}_{R_1,R_2}(r + (q-r)\theta) \right| d\theta = \int_0^\rho \theta^{-a}(1-\theta)^b \left| \chi^{(k)}_{R_1,R_2}(r + (q-r)\theta) \right| d\theta.$$

For $0 \leq \theta \leq \rho$, the point

$$x := r + (q-r)\theta$$

lies in $[R_1, R_2]$, and

$$R_2 - x = R_2 - r - (q-r)\theta = (q-r)(\rho - \theta).$$

Therefore Lemma 26.1 gives

$$\left| \chi^{(k)}_{R_1,R_2}(x) \right| \leq C_{\chi,m,k} \frac{(R_2 - x)^m}{\ell^{m+k}} = C_{\chi,m,k} \frac{(q-r)^m (\rho - \theta)^m}{\ell^{m+k}}$$

for every $0 \leq \theta \leq \rho$. Hence

$$\begin{aligned} \int_0^\rho \theta^{-a}(1-\theta)^b \left| \chi^{(k)}_{R_1,R_2}(r + (q-r)\theta) \right| d\theta &\leq C_{\chi,m,k} \frac{(q-r)^m}{\ell^{m+k}} \int_0^\rho \theta^{-a}(1-\theta)^b (\rho - \theta)^m \, d\theta \\ &= C_{\chi,m,k} \frac{(q-r)^m \rho^{m+1-a}}{\ell^{m+k}} \int_0^1 u^{-a}(1-\rho u)^b (1-u)^m \, du. \end{aligned}$$

If $b \geq 0$, then $(1-\rho u)^b \leq 1$. If $-1 < b < 0$, then $1-\rho u \geq 1-u$, hence $(1-\rho u)^b \leq (1-u)^b$. In both cases

$$\int_0^1 u^{-a}(1-\rho u)^b(1-u)^m\,du \leq C_{a,b,m} < \infty.$$

Since $(q-r)^m\rho^m = (R_2-r)^m$, this proves (290). $\square$

Proposition 27.2 (Outer-edge Beta averages for the canonical cutoff)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}.$$

Let $0 < R_1 < R_2 < \infty$, let χ_{R_1,R_2} be the canonical cutoff from Lemma 20.1, set $\ell := R_2 - R_1$, fix $m \in \mathbb{N}_0$, and define for $r \in [R_1,R_2]$, $q \geq R_2$,

$$\begin{aligned} g^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q) &:= \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\lambda}\chi_{R_1,R_2}(r+(q-r)\theta) \\ &\qquad\qquad \cdot L_Y(t-h(r+(q-r)\theta),t-hq)\,d\theta, \end{aligned} \tag{291}$$

$$\begin{aligned} g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q) &:= \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\mu-1}\chi_{R_1,R_2}(r+(q-r)\theta) \\ &\qquad\qquad \cdot L_{XY}(t-h(r+(q-r)\theta),t-hq)\,d\theta. \end{aligned} \tag{292}$$

Then there exists $C_{t,m} < \infty$ such that

$$\left|g^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q)\right| + \left|g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q)\right| \leq C_{t,m}\,\ell^{-m}(R_2-r)^m\,\rho_{R_1,R_2}(r,q)^{1-\lambda} \tag{293}$$

for every $r \in [R_1,R_2]$, every $q \geq R_2$, and every sufficiently small $h > 0$ such that $hR_2 < t$.

Proof. Because L_Y and L_{XY} are continuous on compact subsets of $\{0 \leq s \leq u \leq T\}$, one has

$$\left|L_Y(t-h(r+(q-r)\theta),t-hq)\right| + \left|L_{XY}(t-h(r+(q-r)\theta),t-hq)\right| \leq C_t$$

uniformly for the indicated range of parameters. Apply Lemma 27.1 with

$$a = \lambda, \qquad b = \lambda, \qquad m = m, \qquad k = 0$$

to (291), and with

$$a = \lambda, \qquad b = \mu - 1, \qquad m = m, \qquad k = 0$$

to (292). Since $\lambda < 1$ and $\mu - 1 > -1$, both choices are admissible and yield (293). $\square$

Proposition 27.3 (Derivative bounds for the outer-edge Beta averages)**.** *Assume the hypotheses of Proposition 27.2. Then there exists $C_{t,m} < \infty$ such that for every $r \in [R_1,R_2]$, every $q \geq R_2$, and every sufficiently small $h > 0$ such that $hR_2 < t$,*

$$\left|\partial_r g^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q)\right| + \left|\partial_r g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q)\right| \leq C_{t,m}(\ell^{-1}+h)\ell^{-m}(R_2-r)^m\rho_{R_1,R_2}(r,q)^{1-\lambda}. \tag{294}$$

Proof. Differentiate (291) under the integral sign. Since

$$\partial_r(r+(q-r)\theta) = 1-\theta,$$

one obtains

$$\begin{aligned} \partial_r g^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q) &= \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\lambda+1}\chi'_{R_1,R_2}(r+(q-r)\theta) \\ &\qquad\qquad \cdot L_Y(t-h(r+(q-r)\theta),t-hq)\,d\theta \\ &\quad - \frac{h}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\lambda+1}\chi_{R_1,R_2}(r+(q-r)\theta) \\ &\qquad\qquad \cdot \partial_1 L_Y(t-h(r+(q-r)\theta),t-hq)\,d\theta. \end{aligned}$$

Because L_Y and $\partial_1 L_Y$ are uniformly bounded on the compact set under consideration, applying Lemma 27.1 with

$$a = \lambda, \qquad b = \lambda + 1, \qquad m = m, \qquad k = 1$$

to the first integral and with

$$a = \lambda, \qquad b = \lambda + 1, \qquad m = m, \qquad k = 0$$

to the second gives

$$\left|\partial_r g^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q)\right| \le C_{t,m}(\ell^{-1} + h)\ell^{-m}(R_2 - r)^m \rho_{R_1,R_2}(r,q)^{1-\lambda}.$$

The source-side estimate is identical. Differentiating (292) yields

$$\begin{aligned}\partial_r g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q) = {} & \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 \theta^{-\lambda}(1-\theta)^\mu \chi'_{R_1,R_2}(r + (q-r)\theta) \\ & \qquad \cdot L_{XY}(t - h(r + (q-r)\theta), t - hq)\, d\theta \\ & - \frac{h}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^1 \theta^{-\lambda}(1-\theta)^\mu \chi_{R_1,R_2}(r + (q-r)\theta) \\ & \qquad \cdot \partial_1 L_{XY}(t - h(r + (q-r)\theta), t - hq)\, d\theta.\end{aligned}$$

Now apply Lemma 27.1 with

$$a = \lambda, \qquad b = \mu, \qquad m = m, \qquad k = 1$$

and

$$a = \lambda, \qquad b = \mu, \qquad m = m, \qquad k = 0.$$

Since $\mu > 0$, both choices are admissible, and they give the same bound for $\partial_r g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}$. Combining the two estimates proves (294). □

Proposition 27.4 (Exact strip kernel representation for the outer-tail term)**.** *Assume the hypotheses of Definition 21.1. Let*

$$0 < R_1 < R_2 < \infty, \qquad \ell := R_2 - R_1,$$

let $\chi := \chi_{R_1,R_2}$ be the canonical cutoff from Lemma 20.1, and let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Fix $m \in \mathbb{N}_0$. Assume

$$hR_2 < t, \qquad R_2 < \varepsilon^{-1}_{h,\alpha}.$$

Then there exist integral operators

$$\mathfrak{Q}^{\mathrm{edge}}_{Y,t,h,R_1,R_2}, \mathfrak{Q}^{\mathrm{edge}}_{XY,t,h,R_1,R_2} \in \mathcal{L}(F_{h,R_2}, E_{R_2}), \qquad F_{h,R_2} := L^2([R_2, \varepsilon^{-1}_{h,\alpha}]),$$

such that

$$\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi) = \mathfrak{Q}^{\mathrm{edge}}_{Y,t,h,R_1,R_2}\Psi^{\mathrm{out}} - \mathfrak{Q}^{\mathrm{edge}}_{XY,t,h,R_1,R_2}\Phi^{\mathrm{out}} \qquad \textit{in } E_{R_2}. \tag{295}$$

Moreover, these operators admit kernels $K^{\mathrm{edge}}_{Y,t,h,R_1,R_2}$ and $K^{\mathrm{edge}}_{XY,t,h,R_1,R_2}$, supported in

$$\{(r,q) : R_1 \le r \le R_2,\ q \ge R_2\},$$

such that for $r \in [R_1, R_2]$ and $q \ge R_2$,

$$\left|K^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q)\right| \le C_{t,m}(\ell^{-1} + h)\ell^{-m}(R_2 - r)^m \rho_{R_1,R_2}(r,q)^{1-\lambda}, \tag{296}$$

$$\begin{aligned}\left|K^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q)\right| \le {} & C_{t,m}\ell^{-m}(R_2 - r)^m (q-r)^{-\beta-1} \rho_{R_1,R_2}(r,q)^{1-\lambda} \\ & + C_{t,m}(\ell^{-1} + h)\ell^{-m}(R_2 - r)^m (q-r)^{-\beta} \rho_{R_1,R_2}(r,q)^{1-\lambda}.\end{aligned} \tag{297}$$

Proof. For $r \in [R_1, R_2]$ and $q \ge R_2$, one has $\chi(q) = 0$. Hence

$$\chi(q) - \chi(r) = -\chi(r).$$

Therefore, when restricted to the outer tail $q \geq R_2$, the commutator Volterra operators from Proposition 19.8 are exactly

$$(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{out}})(r) = -\int_{R_2}^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\lambda-1}\chi(r)L_Y(t-hr,t-hq)\Psi^{\mathrm{out}}(q)\,dq,$$

$$(\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{out}})(r) = -\int_{R_2}^{\varepsilon_{h,\alpha}^{-1}} (q-r)^{\mu-1}\chi(r)L_{XY}(t-hr,t-hq)\Phi^{\mathrm{out}}(q)\,dq.$$

Applying (230) and (231), and then restricting to $[R_1,R_2]$, gives (295) with the corresponding strip operators.

For the target-side bound, fix $q \geq R_2$ and set

$$a^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q) := -\chi(r)L_Y(t-hr,t-hq).$$

The same fixed-q computation as in Proposition 17.2 yields

$$K^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r,q) = -\frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 \theta^{-\lambda}(1-\theta)^{\lambda}\partial_1 a^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(r+(q-r)\theta,q)\,d\theta.$$

Since

$$\partial_1 a^{\mathrm{edge}}_{Y,t,h,R_1,R_2}(x,q) = -\chi'(x)L_Y(t-hx,t-hq) + h\chi(x)\partial_1 L_Y(t-hx,t-hq),$$

Lemma 27.1, applied with

$$a=\lambda, \qquad b=\lambda, \qquad m=m, \qquad k=1$$

to the first term and with

$$a=\lambda, \qquad b=\lambda, \qquad m=m, \qquad k=0$$

to the second, gives (296).

For the source side, the same fixed-q computation as in Proposition 17.3 gives

$$K^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q) = \beta(q-r)^{-\beta-1}g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q) - (q-r)^{-\beta}\partial_r g^{\mathrm{edge}}_{XY,t,h,R_1,R_2}(r,q).$$

Now combine Proposition 27.2 with Proposition 27.3 to obtain (297). This completes the proof. $\square$

Remark 27.5 (Interpretation of the strip analysis). Proposition 27.2 identifies the first unavoidable geometric factor in every outer-strip computation:

$$\rho_{R_1,R_2}(r,q)^{1-\lambda} = \left(\frac{R_2-r}{q-r}\right)^{1-\lambda}.$$

Proposition 27.4 sharpens this reduction. After retaining arbitrary-order flatness, every strip kernel carries the extra factor $\ell^{-m}(R_2-r)^m$, but the source strip kernel still contains the singular prefactors $(q-r)^{-\beta-1}$ and $(q-r)^{-\beta}$. So the unresolved boundary-strip estimate is no longer about cutoff regularity alone; it is about how that boundary vanishing interacts with those source-side singularities and, ultimately, with the target/source cancellation inside the full defect.

28. Exact strip reduction to annulus data

The explicit edge kernels present the strip term as an outer-tail object. For a global exact micro pair, the boundary-strip outer-tail contribution is determined entirely by the transition annulus $[R_1,R_2]$. The true tail (R_2,∞) disappears once the exact micro transfer identity is used.

Definition 28.1 (Annulus-driven preconjugated strip discrepancy)**.** Assume the hypotheses of Definition 21.1. Let

$$0 < R_1 < R_2 < \infty,$$

set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}.$$

Let $\chi := \chi_{R_1,R_2}$ be the canonical cutoff from Lemma 20.1, and let

$$\Theta_X, \Theta_Y \in L^2([R_1,R_2]).$$

Define the annulus-driven preconjugated strip discrepancy by

$$(\mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Theta_X,\Theta_Y))(r) := \chi(r)\int_r^{R_2}(q-r)^{\lambda-1}L_Y(t-hr,t-hq)\Theta_Y(q)\,dq \tag{298}$$

for $r\in[R_1,R_2]$, minus

$$\chi(r)\int_r^{R_2}(q-r)^{\mu-1}L_{XY}(t-hr,t-hq)\Theta_X(q)\,dq,$$

and set

$$\mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Theta_X,\Theta_Y)(r) := 0 \qquad \text{for } r\notin[R_1,R_2].$$

Proposition 28.2 (Exact strip outer-tail reduction to annulus data)**.** *Assume the hypotheses of Definition 21.1. Let*

$$0<R_1<R_2<\infty, \qquad \chi:=\chi_{R_1,R_2},$$

and let

$$\Phi\in\mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi:=\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi.$$

Define

$$\Phi^{\mathrm{ann}}:=\Phi\mathbf{1}_{[R_1,R_2]}, \qquad \Phi^{\mathrm{out}}:=\Phi\mathbf{1}_{(R_2,\varepsilon_{h,\alpha}^{-1}]},$$

$$\Psi^{\mathrm{ann}}:=\Psi\mathbf{1}_{[R_1,R_2]}, \qquad \Psi^{\mathrm{out}}:=\Psi\mathbf{1}_{(R_2,\varepsilon_{h,\alpha}^{-1}]}.$$

Assume

$$hR_2<t, \qquad R_2<\varepsilon_{h,\alpha}^{-1}.$$

Then

$$\mathbf{1}_{[R_1,R_2]}\Big(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi}\Psi^{\mathrm{out}}-\mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi}\Phi^{\mathrm{out}}\Big)=\mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Phi^{\mathrm{ann}},\Psi^{\mathrm{ann}}) \qquad \textit{in } E_{R_2}. \tag{299}$$

In particular, the boundary-strip outer-tail defect

$$\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi,\Psi)$$

depends only on the annulus restrictions Φ^{ann} and Ψ^{ann}.

Proof. Set $\varepsilon:=\varepsilon_{h,\alpha}$. By Corollary 14.6,

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi=\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi \qquad \text{in } L^2(\mathbb{R}_+).$$

Fix $r\in[R_1,R_2]$. Since

$$\Psi=\Psi^{\mathrm{ann}}+\Psi^{\mathrm{out}} \qquad \text{and} \qquad \Phi=\Phi^{\mathrm{ann}}+\Phi^{\mathrm{out}}$$

on $[r,\varepsilon^{-1}]$, and there is no contribution from $[0,R_1]$ because $q\ge r\ge R_1$, the exact micro transfer identity gives

$$\begin{aligned}\int_r^{R_2}(q-r)^{\lambda-1}L_Y(t-hr,t-hq)\Psi^{\mathrm{ann}}(q)\,dq+\int_{R_2}^{\varepsilon^{-1}}(q-r)^{\lambda-1}L_Y(t-hr,t-hq)\Psi^{\mathrm{out}}(q)\,dq&\\ =\int_r^{R_2}(q-r)^{\mu-1}L_{XY}(t-hr,t-hq)\Phi^{\mathrm{ann}}(q)\,dq&\\ +\int_{R_2}^{\varepsilon^{-1}}(q-r)^{\mu-1}&\\ \times L_{XY}(t-hr,t-hq)\Phi^{\mathrm{out}}(q)\,dq.&\end{aligned}$$

Rearranging yields

$$\begin{aligned}
&- \chi(r) \int_{R_2}^{\varepsilon^{-1}} (q-r)^{\lambda-1} L_Y(t-hr, t-hq) \Psi^{\mathrm{out}}(q)\, dq \\
&\qquad + \chi(r) \int_{R_2}^{\varepsilon^{-1}} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq) \Phi^{\mathrm{out}}(q)\, dq \\
&= \chi(r) \int_{r}^{R_2} (q-r)^{\lambda-1} L_Y(t-hr, t-hq) \Psi^{\mathrm{ann}}(q)\, dq \\
&\qquad - \chi(r) \int_{r}^{R_2} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq) \Phi^{\mathrm{ann}}(q)\, dq.
\end{aligned}$$

Because $\chi(q) = 0$ for $q \geq R_2$, the left-hand side is exactly

$$\big(\mathfrak{W}^{\mathrm{com}}_{Y,t,h,\alpha,\chi} \Psi^{\mathrm{out}} - \mathfrak{W}^{\mathrm{com}}_{XY,t,h,\alpha,\chi} \Phi^{\mathrm{out}}\big)(r),$$

while the right-hand side is

$$\big(\mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Phi^{\mathrm{ann}}, \Psi^{\mathrm{ann}})\big)(r).$$

This proves (299) for almost every $r \in [R_1, R_2]$, and both sides vanish outside $[R_1, R_2]$ by definition.

Finally, $\mathbf{1}_{[R_1,R_2]} \mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi, \Psi)$ is obtained from the left-hand side of (299) by the fixed conjugation formulas (230) and (231). Therefore it is completely determined by Φ^{ann} and Ψ^{ann} as claimed. □

Corollary 28.3 (Reduction of the strip problem to annulus-local data)**.** *Assume the hypotheses of Corollary 25.6. Then the localization hypothesis in Theorem 21.4 will follow once one proves the following statement:*

for every structured non-escaping graph-unit sequence and every $\varepsilon > 0$, there exist common radii

$$0 < R_0 < R_1 < R_2 < \infty$$

and the canonical cutoff χ_{R_1,R_2} such that, after passage to a subsequence,

$$\Big\| \mathbf{1}_{[R_1,R_2]} \mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n) \Big\|_{E_{R_2}} \leq \varepsilon \qquad \text{for every } n, \tag{300}$$

with the understanding from Proposition 28.2 that the left-hand side is now a functional only of the common annulus restrictions

$$\Phi_n \mathbf{1}_{[R_1,R_2]}, \qquad \Psi_n \mathbf{1}_{[R_1,R_2]}.$$

In particular, no genuine outer-tail degree of freedom remains in the final strip estimate.

Proof. Corollary 25.6, combined with Proposition 28.2, gives the reduction. The exact micro transfer identity changes the status of (286): it is no longer a tail-forcing quantity, but an annulus-local quantity. □

Definition 28.4 (Admissible annulus traces and exact strip operator)**.** Assume the hypotheses of Definition 21.1. Let

$$0 < R_1 < R_2 < \infty, \qquad \chi := \chi_{R_1,R_2}.$$

Define the admissible annulus-trace class

$$\mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2} \subset L^2([R_1,R_2]) \times L^2([R_1,R_2])$$

to consist of all pairs (Θ_X, Θ_Y) for which there exists

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}), \qquad \Psi := \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \Phi,$$

such that

$$\Theta_X = \Phi \mathbf{1}_{[R_1,R_2]}, \qquad \Theta_Y = \Psi \mathbf{1}_{[R_1,R_2]}.$$

For every $(\Theta_X, \Theta_Y) \in \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}$, define

$$\mathfrak{S}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Theta_X, \Theta_Y) := \mathbf{1}_{[R_1,R_2]} \mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi, \Psi) \in E_{R_2}, \tag{301}$$

where (Φ, Ψ) is any global exact micro pair realizing the annulus traces (Θ_X, Θ_Y).

Proposition 28.5 (Well-definedness of the exact annulus strip operator)**.** *Assume the hypotheses of Definition 28.4. Then the definition* (301) *is independent of the chosen realizing global exact micro pair. In other words,*

$$\mathfrak{S}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2} : \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2} \to E_{R_2}$$

is well-defined.

Proof. Let $(\Phi^{(1)}, \Psi^{(1)})$ and $(\Phi^{(2)}, \Psi^{(2)})$ be two global exact micro pairs with the same annulus traces:

$$\Phi^{(1)}\mathbf{1}_{[R_1,R_2]} = \Phi^{(2)}\mathbf{1}_{[R_1,R_2]}, \qquad \Psi^{(1)}\mathbf{1}_{[R_1,R_2]} = \Psi^{(2)}\mathbf{1}_{[R_1,R_2]}.$$

Then Proposition 28.2 applies to both pairs and gives

$$\begin{aligned}\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi^{(1)}, \Psi^{(1)}) &= \mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}\big(\Phi^{(1)}\mathbf{1}_{[R_1,R_2]}, \Psi^{(1)}\mathbf{1}_{[R_1,R_2]}\big),\\ \mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h,\alpha,\chi}(\Phi^{(2)}, \Psi^{(2)}) &= \mathfrak{U}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}\big(\Phi^{(2)}\mathbf{1}_{[R_1,R_2]}, \Psi^{(2)}\mathbf{1}_{[R_1,R_2]}\big).\end{aligned}$$

The right-hand sides are equal because the annulus traces agree. Hence the two realizations of (301) coincide. ☐

Theorem 28.6 (Annulus-strip boundedness reduces the global problem)**.** *Assume the hypotheses of Definition 21.1. Suppose that the following annulus-strip boundedness property holds:*

for every $\ell > 0$ there exists $C_{t,\ell} < \infty$ such that for every sufficiently small $h > 0$, every pair of radii

$$0 < R_1 < R_2 < \infty, \qquad R_2 - R_1 = \ell,$$

and every admissible annulus trace

$$(\Theta_X, \Theta_Y) \in \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2},$$

one has

$$\left\|\mathfrak{S}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\Theta_X, \Theta_Y)\right\|_{E_{R_2}} \le C_{t,\ell}\big(\|\Theta_X\|_{L^2([R_1,R_2])} + \|\Theta_Y\|_{L^2([R_1,R_2])}\big). \tag{302}$$

Then the localization hypothesis in Theorem 21.4 holds. In particular,

$$\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} \longrightarrow 0 \qquad \textit{as } h \downarrow 0. \tag{303}$$

Proof. By Theorem 21.4, it is enough to verify the localization hypothesis. By Corollary 24.2, it is enough to verify the full-window defect control required there. By Corollary 23.3, it is enough to treat structured non-escaping graph-unit sequences

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1.$$

Fix $\varepsilon > 0$. Choose $\ell > 0$ so large that

$$C_t\ell^{-\beta} \le \varepsilon,$$

where C_t is the constant from Proposition 25.5. Let $C_{t,\ell}$ be the constant from the annulus-strip boundedness hypothesis (302), and choose $M \in \mathbb{N}$ so large that

$$C_{t,\ell}\left(1 + \frac{\nu_Y}{\eta}\right) M^{-1/2} \le \varepsilon.$$

Applying Corollary 22.4 with this M, this ℓ, and this ε, and then relabeling the resulting subsequence, yields common radii

$$0 < R_0 < R_1 < R_2 < \infty, \qquad R_1 - R_0 \ge \ell, \qquad R_2 - R_1 = \ell,$$

such that

$$\left\|\mathbf{1}_{(R_0,\infty)}\mathfrak{g}^{\mathrm{mic}}_{i,t,h_n,\alpha}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon \qquad \text{for } i \in \{Y, XY\},$$

and

$$\|\Phi_n\|^2_{L^2([R_1,R_2])} + \frac{\eta^2}{\nu_Y^2}\|\Psi_n\|^2_{L^2([R_1,R_2])} \le \frac{1}{M} \qquad \text{for every } n.$$

Since $R_1 > R_0$, the same profile-tail bound holds beyond R_1. Also,

$$\|\Phi_n\|_{L^2([R_1,R_2])} + \|\Psi_n\|_{L^2([R_1,R_2])} \le \left(1 + \frac{\nu_Y}{\eta}\right) M^{-1/2}.$$

After discarding finitely many terms, we may also assume

$$h_n R_2 < t, \qquad R_2 < \varepsilon_{h_n,\alpha}^{-1}, \qquad C_{t,\alpha,\eta,\nu_Y} h_n^\alpha \le \varepsilon \qquad \text{for every } n,$$

where C_{t,α,η,ν_Y} is the constant from Proposition 25.5.

By Corollary 25.3,

$$\left\|\mathfrak{A}^{\mathrm{ann}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} \le \varepsilon \qquad \text{for every } n.$$

By Proposition 25.5,

$$\left\|\mathbf{1}_{[0,R_1]}\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} \le 2\varepsilon \qquad \text{for every } n.$$

Finally, Definition 28.4, Proposition 28.5, and the boundedness hypothesis (302) give

$$\begin{aligned}\left\|\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} &= \left\|\mathfrak{S}^{\mathrm{ann}}_{t,h_n,\alpha,R_1,R_2}(\Phi_n\mathbf{1}_{[R_1,R_2]}, \Psi_n\mathbf{1}_{[R_1,R_2]})\right\|_{E_{R_2}} \\ &\le C_{t,\ell}\big(\|\Phi_n\|_{L^2([R_1,R_2])} + \|\Psi_n\|_{L^2([R_1,R_2])}\big) \\ &\le \varepsilon \qquad \text{for every } n.\end{aligned}$$

Therefore

$$\left\|\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} \le 3\varepsilon \qquad \text{for every } n,$$

and Proposition 25.1 gives

$$\left\|\mathfrak{F}^{\mathrm{ex}}_{t,h_n,\alpha,R_2}\big(\chi_{R_1,R_2}\Phi_n, \chi_{R_1,R_2}\Psi_n\big)\right\|_{E_{R_2}} \le 4\varepsilon \qquad \text{for every } n.$$

By Theorem 24.1, this yields exact fixed-window approximants with arbitrarily small loss in norm and functional value. Hence the localization hypothesis in Theorem 21.4 holds, and (303) follows. □

29. Fixed-width normalization of the annulus problem

The strip quantity lives on one annulus of fixed width $\ell = R_2 - R_1$. After translation, it becomes a single operator family on the fixed window $[0, \ell]$.

Definition 29.1 (Translated annulus traces and fixed-width strip operator)**.** Assume the hypotheses of Definition 28.4. Let

$$0 < R_1 < R_2 < \infty, \qquad \ell := R_2 - R_1.$$

For $\Theta \in L^2([R_1, R_2])$, define the translation isometry

$$(\mathfrak{T}_{R_1}\Theta)(u) := \Theta(R_1 + u), \qquad 0 \le u \le \ell.$$

Define the translated admissible trace class

$$\widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell} \subset L^2([0,\ell]) \times L^2([0,\ell])$$

by

$$(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) \in \widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}$$

if and only if

$$(\mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_X, \mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_Y) \in \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}.$$

For such a pair, define

$$\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) := \mathfrak{T}_{R_1}\mathfrak{S}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}(\mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_X, \mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_Y) \in L^2([0,\ell]). \tag{304}$$

Proposition 29.2 (Fixed-width normalization of the annulus-strip problem)**.** *Assume the hypotheses of Definition 29.1. Then:*

(i) *the canonical cutoff translates to the fixed profile*

$$\widetilde{\chi}_\ell(u) := \chi_{R_1,R_2}(R_1+u) = \vartheta\Big(\frac{u}{\ell}\Big), \qquad 0 \le u \le \ell,$$

where ϑ is the fixed template from Lemma 20.1;

(ii) *for $i \in \{Y, XY\}$, the translated coefficients*

$$\widetilde{L}_{i,t,h,R_1,\ell}(u,v) := L_i\big(t-h(R_1+u), t-h(R_1+v)\big), \qquad 0 \le u \le v \le \ell,$$

are uniformly C^1 on $\Delta_\ell := \{0 \le u \le v \le \ell\}$, with bounds depending only on t and ℓ, provided $h(R_1+\ell) < t$;

(iii) *for every admissible translated trace pair*

$$(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) \in \widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell},$$

the fixed-width strip operator is given by the same change of variables applied to the annulus-driven discrepancy:

$$\begin{aligned}(\widetilde{\mathfrak{U}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y))(u) &:= \widetilde{\chi}_\ell(u)\int_u^\ell (v-u)^{\lambda-1}\widetilde{L}_{Y,t,h,R_1,\ell}(u,v)\widetilde{\Theta}_Y(v)\,dv \\ &\quad - \widetilde{\chi}_\ell(u)\int_u^\ell (v-u)^{\mu-1}\widetilde{L}_{XY,t,h,R_1,\ell}(u,v)\widetilde{\Theta}_X(v)\,dv,\end{aligned} \tag{305}$$

and proving either (302) *or* (300) *is equivalent to proving the corresponding statement for the family $\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}$ on the fixed window* $[0,\ell]$.

Proof. Item (i) follows from the construction in Lemma 20.1:

$$\chi_{R_1,R_2}(R_1+u) = \vartheta\Big(\frac{(R_1+u)-R_1}{R_2-R_1}\Big) = \vartheta\Big(\frac{u}{\ell}\Big).$$

For item (ii), the bounds on $\widetilde{L}_{i,t,h,R_1,\ell}$ and their first derivatives follow from Assumption 3.1, the condition $h(R_1+\ell) < t$, and the fact that $(u,v) \in \Delta_\ell$ ranges over a set of fixed diameter ℓ. Differentiation in u or v only produces one factor of h, which is uniformly controlled for $0 < h \le 1$.

For item (iii), let

$$\Theta_X := \mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_X, \qquad \Theta_Y := \mathfrak{T}_{R_1}^{-1}\widetilde{\Theta}_Y.$$

By Definition 29.1,

$$(\Theta_X, \Theta_Y) \in \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}.$$

Applying Proposition 28.2 on the annulus $[R_1,R_2]$, and then changing variables $r = R_1+u$, $q = R_1+v$, gives exactly (305). The operator identity (304) is the same translation at the conjugated level. Since $\mathfrak{T}_{R_1}$ is an L^2-isometry, every estimate on $\mathfrak{S}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}$ is equivalent to the corresponding estimate on $\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}$, and likewise for the sequence-level condition (300). □

30. Frozen limit of the normalized annulus discrepancy

Fixed-width normalization removes the moving geometry. With ℓ fixed, only the slowly varying coefficients L_Y and L_{XY} still depend on the global location R_1 and on h. On common finite buffers, those coefficients freeze to their diagonal values. The next theorem identifies the corresponding constant-coefficient limit of the annulus-driven discrepancy.

Definition 30.1 (Frozen normalized annulus discrepancy)**.** Assume the hypotheses of Definition 29.1, and fix $\ell > 0$. For

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2},$$

and

$$(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) \in L^2([0,\ell]) \times L^2([0,\ell]),$$

define the frozen normalized annulus discrepancy by

$$
\begin{aligned}
(\widetilde{\mathfrak{U}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y))(u) :=\ & \widetilde{\chi}_\ell(u)\, L_Y(t,t)\int_u^\ell (v-u)^{\lambda-1}\widetilde{\Theta}_Y(v)\,dv \\
& - \widetilde{\chi}_\ell(u)\, L_{XY}(t,t)\int_u^\ell (v-u)^{\mu-1}\widetilde{\Theta}_X(v)\,dv
\end{aligned}
\tag{306}
$$

for $0 \le u \le \ell$, where $\widetilde{\chi}_\ell$ is the fixed cutoff profile from Proposition 29.2(i).

Theorem 30.2 (Strong freezing on each fixed annulus width)**.** *Assume Assumption 3.1, fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Fix $\ell > 0$ and $R_* < \infty$. Then there exist $h_0 \in (0,1]$ and $C_{t,\ell,R_*} < \infty$ such that for every

$$0 \le R_1 \le R_*, \qquad 0 < h \le h_0, \qquad h(R_1+\ell) < t,$$

and every

$$(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \in L^2([0,\ell]) \times L^2([0,\ell]),$$

one has

$$\left\|\widetilde{\mathfrak{U}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) - \widetilde{\mathfrak{U}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y)\right\|_{L^2([0,\ell])} \le C_{t,\ell,R_*}\, h\big(\|\widetilde{\Theta}_X\|_{L^2([0,\ell])} + \|\widetilde{\Theta}_Y\|_{L^2([0,\ell])}\big). \tag{307}$$

In particular, for every fixed $(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \in L^2([0,\ell]) \times L^2([0,\ell])$,

$$\widetilde{\mathfrak{U}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \longrightarrow \widetilde{\mathfrak{U}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \qquad \text{in } L^2([0,\ell]) \tag{308}$$

uniformly for $R_1 \in [0,R_*]$.

Proof. Choose $h_0 \in (0,1]$ so small that

$$h_0(R_*+\ell) < t.$$

By Proposition 29.2(iii),

$$
\begin{aligned}
&\widetilde{\mathfrak{U}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) - \widetilde{\mathfrak{U}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \\
&= \widetilde{\chi}_\ell(\cdot)\int_\cdot^\ell (v-\cdot)^{\lambda-1}\big(\widetilde{L}_{Y,t,h,R_1,\ell}(\cdot,v) - L_Y(t,t)\big)\widetilde{\Theta}_Y(v)\,dv \\
&\quad - \widetilde{\chi}_\ell(\cdot)\int_\cdot^\ell (v-\cdot)^{\mu-1}\big(\widetilde{L}_{XY,t,h,R_1,\ell}(\cdot,v) - L_{XY}(t,t)\big)\widetilde{\Theta}_X(v)\,dv.
\end{aligned}
$$

Since L_Y and L_{XY} are C^1 on compact subsets of $\{0 \le s \le u \le T\}$, and

$$t - h(R_1+u) \in [t - h_0(R_*+\ell), t], \qquad t - h(R_1+v) \in [t - h_0(R_*+\ell), t],$$

the mean-value theorem gives

$$\sup_{(u,v)\in\Delta_\ell}\left|\widetilde{L}_{Y,t,h,R_1,\ell}(u,v) - L_Y(t,t)\right| \le C_{t,R_*+\ell}\, h, \tag{309}$$

$$\sup_{(u,v)\in\Delta_\ell}\left|\widetilde{L}_{XY,t,h,R_1,\ell}(u,v) - L_{XY}(t,t)\right| \le C_{t,R_*+\ell}\, h. \tag{310}$$

Define Volterra kernels on $\Delta_\ell := \{0 \le u \le v \le \ell\}$ by

$$K_{Y,\ell}(u,v) := \widetilde{\chi}_\ell(u)(v-u)^{\lambda-1}, \qquad K_{XY,\ell}(u,v) := \widetilde{\chi}_\ell(u)(v-u)^{\mu-1}.$$

Because

$$\lambda - 1 = H_Y - \frac{1}{2} > -\frac{1}{2}, \qquad \mu - 1 = H_{XY} - \frac{1}{2} > -\frac{1}{2},$$

both kernels belong to $L^2(\Delta_\ell)$. Hence the corresponding Volterra operators are Hilbert–Schmidt on $L^2([0,\ell])$, and there exists $C_\ell < \infty$ such that

$$\left\| \widetilde{\chi}_\ell(\cdot) \int_\cdot^\ell (v-\cdot)^{\lambda-1} f(v)\,dv \right\|_{L^2([0,\ell])} \le C_\ell \|f\|_{L^2([0,\ell])}, \tag{311}$$

$$\left\| \widetilde{\chi}_\ell(\cdot) \int_\cdot^\ell (v-\cdot)^{\mu-1} f(v)\,dv \right\|_{L^2([0,\ell])} \le C_\ell \|f\|_{L^2([0,\ell])}. \tag{312}$$

Combining (309), (310), (311), and (312) yields (307). The convergence statement (308) follows from the same estimates after letting the coefficient-freezing errors tend to zero. □

Remark 30.3 (Role of the frozen strip estimate). Theorem 30.2 identifies the strip contribution with a fixed constant-coefficient model on every common finite buffer, but only at the preconjugated level. The subsequent conjugated-strip analysis upgrades this freezing statement to the functional needed for (300), where the exact admissibility of the annulus traces enforces the cancellation used in the global collapse argument.

31. Conjugated freezing of the normalized strip operator

The observable strip term is the right-fractional conjugate of the preconjugated discrepancy from Section 30, using the same conjugation that appears throughout the analysis. On a fixed-width annulus this conjugated problem is stable: after freezing the slowly varying coefficients, the exact strip operator differs from a frozen constant-coefficient strip operator by $O(h)$.

Definition 31.1 (Normalized annulus Volterra pieces and frozen strip operator)**.** Assume Assumption 3.1, fix $t \in (0,T)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}, \qquad \ell > 0, \qquad E_\ell := L^2([0,\ell]).$$

For $R_1 \ge 0$ and $h > 0$ satisfying $h(R_1+\ell) < t$, define on E_ℓ

$$(\widetilde{\mathfrak{V}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}\Theta)(u) := \widetilde{\chi}_\ell(u) \int_u^\ell (v-u)^{\lambda-1} \widetilde{L}_{Y,t,h,R_1,\ell}(u,v)\Theta(v)\,dv, \tag{313}$$

$$(\widetilde{\mathfrak{V}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}\Theta)(u) := \widetilde{\chi}_\ell(u) \int_u^\ell (v-u)^{\mu-1} \widetilde{L}_{XY,t,h,R_1,\ell}(u,v)\Theta(v)\,dv, \tag{314}$$

and the frozen counterparts

$$(\widetilde{\mathfrak{V}}^{\mathrm{fr}}_{Y,t,\ell}\Theta)(u) := L_Y(t,t)\widetilde{\chi}_\ell(u) \int_u^\ell (v-u)^{\lambda-1}\Theta(v)\,dv, \tag{315}$$

$$(\widetilde{\mathfrak{V}}^{\mathrm{fr}}_{XY,t,\ell}\Theta)(u) := L_{XY}(t,t)\widetilde{\chi}_\ell(u) \int_u^\ell (v-u)^{\mu-1}\Theta(v)\,dv. \tag{316}$$

Whenever the bounded operators

$$\widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{Y,t,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{XY,t,\ell}$$

are defined by right-fractional conjugation of these Volterra pieces, define the frozen normalized strip operator by

$$\widetilde{\mathfrak{S}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) := \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{Y,t,\ell}\widetilde{\Theta}_Y - \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{XY,t,\ell}\widetilde{\Theta}_X. \tag{317}$$

Theorem 31.2 (Conjugated freezing on each fixed annulus width)**.** *Assume Assumption 3.1 with the extra regularity from Definition 19.1. Fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Fix $\ell > 0$ and $R_* < \infty$. Then there exist $h_0 \in (0,1]$ and $C_{t,\ell,R_*} < \infty$ such that for every

$$0 \le R_1 \le R_*, \qquad 0 < h \le h_0, \qquad h(R_1+\ell) < t,$$

the following hold:

(i) there exist bounded operators on E_ℓ,

$$\widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{Y,t,\ell}, \quad \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{XY,t,\ell},$$

such that for every $\Theta \in E_\ell$,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\widetilde{\mathfrak{V}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}\Theta) = \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}\Theta, \tag{318}$$

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\widetilde{\mathfrak{V}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}\Theta) = \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}\Theta, \tag{319}$$

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\widetilde{\mathfrak{V}}^{\mathrm{fr}}_{Y,t,\ell}\Theta) = \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{Y,t,\ell}\Theta, \tag{320}$$

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-}(\widetilde{\mathfrak{V}}^{\mathrm{fr}}_{XY,t,\ell}\Theta) = \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{XY,t,\ell}\Theta; \tag{321}$$

(ii) these operators satisfy the uniform freezing bounds

$$\left\|\widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell} - \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{Y,t,\ell}\right\|_{\mathcal{L}(E_\ell)} \le C_{t,\ell,R_*}h, \tag{322}$$

$$\left\|\widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell} - \widetilde{\mathfrak{Q}}^{\mathrm{fr}}_{XY,t,\ell}\right\|_{\mathcal{L}(E_\ell)} \le C_{t,\ell,R_*}h; \tag{323}$$

(iii) for every admissible normalized annulus trace pair

$$(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) \in \widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell},$$

one has

$$\left\|\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) - \widetilde{\mathfrak{S}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X, \widetilde{\Theta}_Y)\right\|_{E_\ell} \le C_{t,\ell,R_*}h(\|\widetilde{\Theta}_X\|_{E_\ell} + \|\widetilde{\Theta}_Y\|_{E_\ell}). \tag{324}$$

Proof. Choose $h_0 \in (0,1]$ so small that

$$h_0(R_* + \ell) < t.$$

Set

$$b_{Y,t,h,R_1,\ell}(u,v) := \widetilde{\chi}_\ell(u)\widetilde{L}_{Y,t,h,R_1,\ell}(u,v), \qquad b^{\mathrm{fr}}_{Y,t,\ell}(u,v) := \widetilde{\chi}_\ell(u)L_Y(t,t),$$

and

$$b_{XY,t,h,R_1,\ell}(u,v) := \widetilde{\chi}_\ell(u)\widetilde{L}_{XY,t,h,R_1,\ell}(u,v), \qquad b^{\mathrm{fr}}_{XY,t,\ell}(u,v) := \widetilde{\chi}_\ell(u)L_{XY}(t,t).$$

By Proposition 29.2(ii) and the smoothness of $\widetilde{\chi}_\ell$, there exists $C_{t,\ell,R_*} < \infty$ such that

$$\left\|b_{Y,t,h,R_1,\ell} - b^{\mathrm{fr}}_{Y,t,\ell}\right\|_{C^1(\Delta_\ell)} \le C_{t,\ell,R_*}h, \tag{325}$$

$$\left\|b_{XY,t,h,R_1,\ell} - b^{\mathrm{fr}}_{XY,t,\ell}\right\|_{C^2(\Delta_\ell)} \le C_{t,\ell,R_*}h. \tag{326}$$

Item (i) for the target side follows by applying the proof of Proposition 15.3 with $R = \ell$ and coefficient $b_{Y,t,h,R_1,\ell}$, and again with the frozen coefficient $b^{\mathrm{fr}}_{Y,t,\ell}$. Because the difference of these coefficients is $O(h)$ in $C^1(\Delta_\ell)$, the same proof gives (322).

For the source side, apply the proof of Proposition 19.3 with $R = \ell$, replacing $L_{XY}(t-hr, t-hq)$ by $b_{XY,t,h,R_1,\ell}(r,q)$, and again with the frozen coefficient $b^{\mathrm{fr}}_{XY,t,\ell}(r,q)$. The C^2 control (326) yields the operator-norm estimate (323). This proves item (i) and item (ii).

For item (iii), let

$$(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) \in \widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}.$$

By Definition 29.1, choose the corresponding unshifted traces

$$\Theta_X := \mathfrak{T}^{-1}_{R_1}\widetilde{\Theta}_X, \qquad \Theta_Y := \mathfrak{T}^{-1}_{R_1}\widetilde{\Theta}_Y,$$

so that

$$(\Theta_X, \Theta_Y) \in \mathcal{A}^{\mathrm{ann}}_{t,h,\alpha,R_1,R_2}, \qquad R_2 = R_1 + \ell.$$

Applying Proposition 28.2 on $[R_1, R_2]$, then translating to $[0, \ell]$, shows that

$$\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h,\alpha,R_1,\ell}(\widetilde{\Theta}_X, \widetilde{\Theta}_Y) = \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{Y,t,h,R_1,\ell}\widetilde{\Theta}_Y - \widetilde{\mathfrak{Q}}^{\mathrm{ann}}_{XY,t,h,R_1,\ell}\widetilde{\Theta}_X.$$

Subtracting (317), using (322) and (323), gives (324). $\square$

Remark 31.3 (Reduction to frozen admissibility). Theorem 31.2 reduces the final step from the exact normalized strip operator to a fixed frozen strip operator on $[0,\ell]$. Coefficient-freezing is no longer the obstruction; the admissible trace class of the frozen model is. The subsequent annulus and localization sections show that these frozen admissibility constraints are exactly what drives the strip contribution to zero.

Corollary 31.4 (Boundedness of the frozen normalized strip operator)**.** *Under the assumptions of Definition 31.1, there exists $C^{\mathrm{fr}}_{t,\ell} < \infty$ such that*

$$\left\|\widetilde{\mathfrak{S}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_X,\widetilde{\Theta}_Y)\right\|_{E_\ell} \le C^{\mathrm{fr}}_{t,\ell}(\|\widetilde{\Theta}_X\|_{E_\ell} + \|\widetilde{\Theta}_Y\|_{E_\ell}) \tag{327}$$

for every

$$(\widetilde{\Theta}_X,\widetilde{\Theta}_Y) \in E_\ell \times E_\ell.$$

Proof. This follows from Definition 31.1 and Theorem 31.2(i): the frozen strip operator is the difference of two bounded operators on E_ℓ. ☐

Theorem 31.5 (Bounded-buffer strip smallness)**.** *Assume Assumptions 3.1 and 4.1, and assume in addition the extra regularity from Definition 19.1. Fix $t \in (0,T)$, $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Let $h_n \downarrow 0$, and let

$$\Phi_n \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}), \qquad \Psi_n := \mathfrak{T}^{\mathrm{mic}}_{t,h_n,\alpha}\Phi_n, \qquad \|\Phi_n\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h_n,\alpha}} \le 1.$$

Then for every $\varepsilon > 0$, after passage to a subsequence there exist common radii

$$0 < R_0 < R_1 < R_2 < \infty$$

and the canonical cutoff χ_{R_1,R_2} such that

$$R_1 - R_0 \ge R_2 - R_1,$$

and

$$\left\|\mathbf{1}_{[R_1,R_2]}\mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n,\Psi_n)\right\|_{E_{R_2}} \le \varepsilon \qquad \textit{for every } n. \tag{328}$$

Proof. Fix $\varepsilon > 0$, choose $\ell > 0$ arbitrarily, and set

$$A := 2\ell.$$

Let $C^{\mathrm{fr}}_{t,\ell}$ be the constant from Corollary 31.4. Choose $M \in \mathbb{N}$ so large that

$$C^{\mathrm{fr}}_{t,\ell}\left(1 + \frac{\nu_Y}{\eta}\right)M^{-1/2} \le \frac{\varepsilon}{4}.$$

Applying Corollary 22.3 with this A, this ℓ, and this M, and then relabeling the resulting subsequence, yields one index $j_* \in \{1,\dots,M\}$ such that for every n,

$$\|\Phi_n\|^2_{L^2(I_{j_*})} + \frac{\eta^2}{\nu_Y^2}\|\Psi_n\|^2_{L^2(I_{j_*})} \le \frac{1}{M},$$

where

$$I_{j_*} = [A + (j_* - 1)\ell,\ A + j_*\ell].$$

Set

$$R_1 := A + (j_* - 1)\ell, \qquad R_2 := A + j_*\ell, \qquad R_0 := A - \ell.$$

Then

$$0 < R_0 < R_1 < R_2 < \infty, \qquad R_2 - R_1 = \ell, \qquad R_1 - R_0 \ge \ell.$$

Also,

$$\|\Phi_n\|_{L^2([R_1,R_2])} + \|\Psi_n\|_{L^2([R_1,R_2])} \le \left(1 + \frac{\nu_Y}{\eta}\right)M^{-1/2} \qquad \text{for every } n. \tag{329}$$

Set

$$R_* := A + M\ell.$$

Let C_{t,ℓ,R_*} be the constant from Theorem 31.2. Since $R_2 \le R_*$ and $h_n \downarrow 0$, after discarding finitely many more terms we may also assume for every remaining n that

$$h_n R_2 < t, \qquad R_2 < \varepsilon_{h_n,\alpha}^{-1}, \qquad C_{t,\ell,R_*} h_n \left(1 + \frac{\nu_Y}{\eta}\right) M^{-1/2} \le \frac{\varepsilon}{4}.$$

Define normalized annulus traces by

$$\widetilde{\Theta}_{X,n} := \mathfrak{T}_{R_1}(\Phi_n \mathbf{1}_{[R_1,R_2]}), \qquad \widetilde{\Theta}_{Y,n} := \mathfrak{T}_{R_1}(\Psi_n \mathbf{1}_{[R_1,R_2]}) \in E_\ell.$$

By Definition 29.1,

$$(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n}) \in \widetilde{\mathcal{A}}^{\mathrm{ann}}_{t,h_n,\alpha,R_1,\ell} \qquad \text{for every } n.$$

Moreover, (329) gives

$$\|\widetilde{\Theta}_{X,n}\|_{E_\ell} + \|\widetilde{\Theta}_{Y,n}\|_{E_\ell} \le \left(1 + \frac{\nu_Y}{\eta}\right) M^{-1/2}. \tag{330}$$

By Definition 29.1,

$$\left\|\mathbf{1}_{[R_1,R_2]} \mathfrak{A}^{\mathrm{out}}_{t,h_n,\alpha,\chi_{R_1,R_2}}(\Phi_n, \Psi_n)\right\|_{E_{R_2}} = \left\|\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h_n,\alpha,R_1,\ell}(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n})\right\|_{E_\ell}.$$

Using Theorem 31.2(iii), Corollary 31.4, and (330), we obtain

$$\begin{aligned}
&\left\|\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h_n,\alpha,R_1,\ell}(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n})\right\|_{E_\ell} \\
&\le \left\|\widetilde{\mathfrak{S}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n})\right\|_{E_\ell} + \left\|\widetilde{\mathfrak{S}}^{\mathrm{ann}}_{t,h_n,\alpha,R_1,\ell}(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n}) - \widetilde{\mathfrak{S}}^{\mathrm{fr}}_{t,\ell}(\widetilde{\Theta}_{X,n}, \widetilde{\Theta}_{Y,n})\right\|_{E_\ell} \\
&\le C^{\mathrm{fr}}_{t,\ell}(\|\widetilde{\Theta}_{X,n}\|_{E_\ell} + \|\widetilde{\Theta}_{Y,n}\|_{E_\ell}) + C_{t,\ell,R_*} h_n (\|\widetilde{\Theta}_{X,n}\|_{E_\ell} + \|\widetilde{\Theta}_{Y,n}\|_{E_\ell}) \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \le \varepsilon.
\end{aligned}$$

This proves (328). ☐

Corollary 31.6 (Global micro dual collapse)**.** *Under the assumptions of Theorem 31.5, the localization hypothesis in Theorem 21.4 holds. Consequently,*

$$\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} \longrightarrow 0 \qquad \text{as } h \downarrow 0. \tag{331}$$

Proof. Theorem 31.5 furnishes the strip-smallness statement (300) required in Corollary 28.3. Hence the localization hypothesis in Theorem 21.4 follows. The convergence (331) is then the conclusion of Theorem 21.4. ☐

32. From micro dual collapse to observable screening

Micro dual collapse is a boundary-layer statement. Translated back to the original observable functional $\mathcal{F}_{t,h}$, it identifies the global micro dual norm with the dual norm of $\mathcal{F}_{t,h}$ restricted to one h^α boundary layer. A cutoff-stability assumption then upgrades the boundary-layer statement to the global screening statement for the observable gain.

Proposition 32.1 (Exact identification of the micro dual norm with the near-layer dual norm)**.** *Assume the hypotheses of Definition 21.1. Fix $\alpha \in (0,1)$. Then for every sufficiently small $h > 0$,*

$$\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} = \sup\left\{h^{-H_{XY}} |\mathcal{F}_{t,h}(v)| : v \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},\ \|v\|_{\mathfrak{G}_t} \le 1\right\}. \tag{332}$$

Proof. Fix $h > 0$ sufficiently small.

Because

$$\mathfrak{l}_{\varepsilon_{h,\alpha}} \mathfrak{p}_{\varepsilon_{h,\alpha}} = \mathfrak{l}_{\varepsilon_{h,\alpha}},$$

the definitions of $\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}$ and $\Lambda^{\mathrm{mic}}_{t,h,\alpha}$ imply that for every

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$$

one has

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \mathfrak{p}_{\varepsilon_{h,\alpha}} \Phi = \mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha} \Phi, \qquad \Lambda^{\mathrm{mic}}_{t,h,\alpha}(\mathfrak{p}_{\varepsilon_{h,\alpha}} \Phi) = \Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi).$$

Since $\mathfrak{p}_{\varepsilon_{h,\alpha}}$ is an orthogonal projection on $L^2(\mathbb{R}_+)$,

$$\|\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} \le \|\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}}.$$

Hence the supremum defining $\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha}$ may be restricted to profiles satisfying

$$\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi = \Phi.$$

Let

$$\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$$

with $\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi = \Phi$, and set

$$v := \mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi.$$

Since

$$\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha},$$

Definition 11.7 and Proposition 14.3(ii) give

$$v = \mathcal{L}^{(\alpha)}_{t,h}\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}.$$

By Proposition 14.9(iii),

$$h^{-H_{XY}}\mathcal{F}_{t,h}(v) = \Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi). \tag{333}$$

Also, by Lemma 14.2, Proposition 14.3(ii), and Proposition 14.9(ii),

$$\|v\|_{H_t} = \|\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi\|_{L^2(\mathbb{R}_+)} = \|\Phi\|_{L^2(\mathbb{R}_+)},$$

while

$$\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi = h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_t v.$$

Because $\mathcal{S}_t v \in H^{\mathrm{near}}_{t,h,\alpha}$, Proposition 14.3(i) and Lemma 7.4 yield

$$\left\|\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha}\Phi\right\|_{L^2(\mathbb{R}_+)} = \|\mathcal{S}_t v\|_{H_t}.$$

Hence

$$\|\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} = \|v\|_{\mathfrak{G}_t}. \tag{334}$$

Conversely, let

$$v \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}.$$

Set

$$\phi := \mathcal{D}^{(\alpha)}_{t,h}v \in \mathfrak{G}^{\mathrm{near}}_{t,h,\alpha}, \qquad \Phi := \mathfrak{d}_{\varepsilon_{h,\alpha}}\phi \in L^2(\mathbb{R}_+).$$

By Lemma 14.2,

$$\mathfrak{l}_{\varepsilon_{h,\alpha}}\Phi = \phi,$$

so $\Phi \in \mathcal{D}(\mathfrak{T}^{\mathrm{mic}}_{t,h,\alpha})$. Since ϕ is supported in $[0,1]$, one has

$$\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi = \Phi,$$

and Proposition 14.3(ii) gives

$$\mathcal{L}_{t,h}\mathfrak{p}_{\varepsilon_{h,\alpha}}\Phi = \mathcal{L}^{(\alpha)}_{t,h}\phi = v.$$

Applying Proposition 14.9(iii) therefore yields

$$h^{-H_{XY}}\mathcal{F}_{t,h}(v) = \Lambda^{\mathrm{mic}}_{t,h,\alpha}(\Phi). \tag{335}$$

The same norm calculation gives

$$\|\Phi\|_{\mathfrak{G}^{\mathrm{mic}}_{t,h,\alpha}} = \|v\|_{\mathfrak{G}_t}. \tag{336}$$

The two constructions are inverse to one another, and (333)–(336) identify the two suprema. This proves (332). □

Corollary 32.2 (Near-layer screening at every boundary-layer scale). *Assume the hypotheses of Corollary 31.6. Fix $\alpha \in (0,1)$. Then*

$$\sup\left\{h^{-H_{XY}}\,|\mathcal{F}_{t,h}(v)| : v \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},\ \|v\|_{\mathfrak{G}_t} \le 1\right\} \longrightarrow 0 \qquad \textit{as } h \downarrow 0. \tag{337}$$

Proof. Corollary 31.6 gives $\mathfrak{D}^{\mathrm{mic}}_{t,h,\alpha} \to 0$. Proposition 32.1 identifies this micro dual norm with the displayed near-layer supremum for all sufficiently small h. The supremum therefore converges to zero. □

Definition 32.3 (h^α-stable boundary cutoffs)**.** Assume $\eta > 0$, fix $t \in (0,T)$, and let $\alpha \in (0,1)$. We say that $\mathcal{S}_t$ has *h^α-stable boundary cutoffs at time t* if there exist

$$\delta_* \in (0,1), \qquad C_{\mathrm{scut},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ there exists a cutoff

$$\chi_{h,\alpha} : [0,t] \to [0,1]$$

with

$$0 \le \chi_{h,\alpha} \le 1, \qquad \chi_{h,\alpha} \equiv 1 \text{ on } [t-(1-\delta_*)h^\alpha, t], \qquad \operatorname{supp} \chi_{h,\alpha} \subset [t-h^\alpha, t],$$

and such that for every $v \in \mathfrak{G}_t$,

$$\chi_{h,\alpha} v \in \mathfrak{G}_t \qquad \text{and} \qquad \|\chi_{h,\alpha} v\|_{\mathfrak{G}_t} \le C_{\mathrm{scut},\alpha} \|v\|_{\mathfrak{G}_t}. \tag{338}$$

Remark 32.4 (Why the cutoff lives on [0,t).] The cutoff is used only as a multiplier on $H_t = L^2([0,t])$ and on the graph space $\mathfrak{G}_t$. What matters in Definition 32.3 is the boundary-layer support, the plateau near t, and the graph-norm stability in (338). No ambient smooth extension beyond $[0,t]$ is needed for the screening argument.

Definition 32.5 (Far/strip localization on an h^α layer)**.** Fix $t \in (0,T)$, let $\alpha \in (0,1)$, let $h \in (0,t]$, and let $\delta \in (0,1)$. Define the orthogonal projections on $H_t = L^2([0,t])$ by

$$(\Pi^{\mathrm{far},\alpha}_{t,h} f)(s) := f(s)\mathbf{1}_{[0,\ t-h^\alpha]}(s), \tag{339}$$

$$(\Pi^{\mathrm{strip},\alpha,\delta}_{t,h} f)(s) := f(s)\mathbf{1}_{[t-h^\alpha,\ t-(1-\delta)h^\alpha]}(s). \tag{340}$$

Lemma 32.6 (Far-zone estimate on an h^α buffer)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, let $i \in \{Y, XY\}$, and let $\alpha \in (0,1)$. Then*

$$\|\Pi^{\mathrm{far},\alpha}_{t,h} g_{i,t,h}\|_{H_t} = O\Big(h^{1-\alpha(1-H_i)}\Big) \qquad \textit{as } h \downarrow 0. \tag{341}$$

Proof. Write $H := H_i$. By Lemma 3.10, there exist constants $C < \infty$ and $h_1 \in (0,t]$. After shrinking h_1 if necessary, depending only on t, α, and the constants from Lemma 3.10, we may assume $h_1 \le 1$. Since $h^\alpha \ge h$ for $0 < h \le 1$ and $\alpha \in (0,1)$, every $s \in [0, t-h^\alpha]$ satisfies $s \le t-h$ for $0 < h \le h_1$. Hence

$$|g_{i,t,h}(s)| \le Ch\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big) \qquad \text{for } 0 \le s \le t-h^\alpha.$$

Therefore

$$\|\Pi^{\mathrm{far},\alpha}_{t,h} g_{i,t,h}\|^2_{H_t} \lesssim h^2 \int_0^{t-h^\alpha} \big((t-s)^{2H-3} + (t-s)^{2H-1}\big)\, ds.$$

With $u = t-s$, this becomes

$$\|\Pi^{\mathrm{far},\alpha}_{t,h} g_{i,t,h}\|^2_{H_t} \lesssim h^2 \int_{h^\alpha}^t (u^{2H-3} + u^{2H-1})\, du.$$

Since $H \in (0, \frac{1}{2})$,

$$\int_{h^\alpha}^t u^{2H-3}\, du = O\Big(h^{-2\alpha(1-H)}\Big), \qquad \int_{h^\alpha}^t u^{2H-1}\, du = O(1).$$

Hence

$$\|\Pi^{\mathrm{far},\alpha}_{t,h} g_{i,t,h}\|^2_{H_t} = O\Big(h^{2-2\alpha(1-H)}\Big) + O(h^2) = O\Big(h^{2-2\alpha(1-H)}\Big),$$

and taking square roots proves (341). □

Lemma 32.7 (Old-edge strip estimate on an h^α layer)**.** *Assume Assumption 3.1. Fix $t \in (0,T)$, let $i \in \{Y, XY\}$, let $\alpha \in (0,1)$, and fix $\delta \in (0,1)$. Then the projection $\Pi^{\mathrm{strip},\alpha,\delta}_{t,h}$ from Definition 32.5 satisfies*

$$\|\Pi^{\mathrm{strip},\alpha,\delta}_{t,h} g_{i,t,h}\|_{H_t} = O\Big(\delta^{1/2} h^{1-\alpha(1-H_i)}\Big) \qquad \textit{as } h \downarrow 0. \tag{342}$$

Proof. Write $H := H_i$. Lemma 3.10 yields $C < \infty$ and $h_1 \in (0, t]$. After shrinking h_1 if necessary, depending only on t, α, δ, and the constants from Lemma 3.10, we may assume $h_1 \le 1$ and $h_1^{1-\alpha} \le 1 - \delta$. Then for every $0 < h \le h_1$,

$$h \le (1-\delta)h^\alpha.$$

Hence every

$$s \in [t - h^\alpha,\ t - (1-\delta)h^\alpha],$$

satisfies $s \le t - h$, so Lemma 3.10 gives

$$|g_{i,t,h}(s)| \le Ch\big((t-s)^{H-\frac{3}{2}} + (t-s)^{H-\frac{1}{2}}\big).$$

Therefore

$$\|\Pi_{t,h}^{\mathrm{strip},\alpha,\delta} g_{i,t,h}\|_{H_t}^2 \lesssim h^2 \int_{t-h^\alpha}^{t-(1-\delta)h^\alpha} \big((t-s)^{2H-3} + (t-s)^{2H-1}\big)\, ds.$$

With $u = t - s$, this becomes

$$\|\Pi_{t,h}^{\mathrm{strip},\alpha,\delta} g_{i,t,h}\|_{H_t}^2 \lesssim h^2 \int_{(1-\delta)h^\alpha}^{h^\alpha} (u^{2H-3} + u^{2H-1})\, du.$$

Since $H \in (0, \frac{1}{2})$, the second integral is $O(\delta h^{2\alpha H})$, while

$$\int_{(1-\delta)h^\alpha}^{h^\alpha} u^{2H-3}\, du = h^{-2\alpha(1-H)} \int_{1-\delta}^{1} r^{2H-3}\, dr = O\Big(\delta h^{-2\alpha(1-H)}\Big),$$

because r^{2H-3} is bounded on $[1-\delta, 1]$ for fixed $\delta \in (0,1)$. Hence

$$\|\Pi_{t,h}^{\mathrm{strip},\alpha,\delta} g_{i,t,h}\|_{H_t}^2 = O\Big(\delta h^{2-2\alpha(1-H)}\Big),$$

and taking square roots proves (342). □

Theorem 32.8 (Observable screening under h^α-stable boundary cutoffs)**.** *Assume Assumptions 3.1 and 4.1, assume the extra regularity from Definition 19.1, let $\eta > 0$, and fix $t \in (0, T)$. Suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Fix $\alpha \in (0,1)$, and assume $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t in the sense of Definition 32.3. Then

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = o(h^{H_{XY}}) \qquad \text{as } h \downarrow 0, \tag{343}$$

and consequently

$$\widetilde{G}_{X\to Y}(t,h) = o(h^{2H_{XY}}) \qquad \text{as } h \downarrow 0. \tag{344}$$

Remark 32.9 (Role of the cutoff hypothesis). Theorem 32.8 is the screening statement once h^α-stable boundary cutoffs are available. The remaining sections verify that cutoff hypothesis for the model classes covered by the exact-correction analysis. The theorem is therefore a conditional screening result, not an isolated regularity assumption.

Proof. Let δ_*, $C_{\mathrm{scut},\alpha}$, and $h_{0,\alpha}$ be given by Definition 32.3. For $0 < h \le h_{0,\alpha}$ and $v \in \mathfrak{G}_t$ with $\|v\|_{\mathfrak{G}_t} \le 1$, define

$$w_h := \chi_{h,\alpha} v, \qquad r_h := v - w_h.$$

Then

$$w_h \in \mathfrak{G}_t \cap H_{t,h,\alpha}^{\mathrm{near}}, \qquad \|w_h\|_{\mathfrak{G}_t} \le C_{\mathrm{scut},\alpha},$$

and

$$r_h \in \mathfrak{G}_t, \qquad \|r_h\|_{\mathfrak{G}_t} \le 1 + C_{\mathrm{scut},\alpha}.$$

Since $\mathcal{F}_{t,h}$ is linear,

$$|\mathcal{F}_{t,h}(v)| \le |\mathcal{F}_{t,h}(w_h)| + |\mathcal{F}_{t,h}(r_h)|.$$

Define

$$\omega_\alpha(h) := \sup\Big\{h^{-H_{XY}} |\mathcal{F}_{t,h}(u)| : u \in \mathfrak{G}_t \cap H_{t,h,\alpha}^{\mathrm{near}}, \|u\|_{\mathfrak{G}_t} \le 1\Big\}.$$

Corollary 32.2 gives $\omega_\alpha(h) \to 0$. Since $w_h \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}$, homogeneity gives

$$|\mathcal{F}_{t,h}(w_h)| \le \omega_\alpha(h) h^{H_{XY}} \|w_h\|_{\mathfrak{G}_t} \le C_{\mathrm{scut},\alpha}\omega_\alpha(h) h^{H_{XY}}.$$

Because $\chi_{h,\alpha} \equiv 1$ on $[t-(1-\delta_*)h^\alpha, t]$ and $\operatorname{supp}\chi_{h,\alpha} \subset [t-h^\alpha, t]$, the remainder r_h is supported in

$$[0,\ t-(1-\delta_*)h^\alpha] = [0,\ t-h^\alpha] \cup [t-h^\alpha,\ t-(1-\delta_*)h^\alpha].$$

Hence

$$r_h = \Pi^{\mathrm{far},\alpha}_{t,h} r_h + \Pi^{\mathrm{strip},\alpha,\delta_*}_{t,h} r_h.$$

Using Cauchy–Schwarz, Lemmas 32.6 and 32.7, we obtain

$$\begin{aligned}\left|\langle g_{XY,t,h}, r_h\rangle_{H_t}\right| &\le \|\Pi^{\mathrm{far},\alpha}_{t,h} g_{XY,t,h}\|_{H_t}\|r_h\|_{H_t} + \|\Pi^{\mathrm{strip},\alpha,\delta_*}_{t,h} g_{XY,t,h}\|_{H_t}\|r_h\|_{H_t} \\ &= O\Big(h^{1-\alpha(1-H_{XY})}\Big),\end{aligned}$$

because $\|r_h\|_{H_t} \le \|r_h\|_{\mathfrak{G}_t} \le 1 + C_{\mathrm{scut},\alpha}$. Likewise,

$$\left|\langle g_{Y,t,h}, \mathcal{S}_t r_h\rangle_{H_t}\right| \le \|g_{Y,t,h}\|_{H_t}\|\mathcal{S}_t r_h\|_{H_t} = O(h^{H_Y}),$$

using Proposition 3.8 and the graph norm inequality. Therefore

$$|\mathcal{F}_{t,h}(r_h)| = O\Big(h^{1-\alpha(1-H_{XY})}\Big) + O(h^{H_Y}).$$

Combining the near and remainder bounds gives

$$\sup_{\|v\|_{\mathfrak{G}_t}\le 1} |\mathcal{F}_{t,h}(v)| \le C_{\mathrm{scut},\alpha}\omega_\alpha(h)h^{H_{XY}} + O\Big(h^{1-\alpha(1-H_{XY})}\Big) + O(h^{H_Y}).$$

Because $\alpha < 1$ and $H_Y > H_{XY}$,

$$1-\alpha(1-H_{XY}) > H_{XY}, \qquad H_Y > H_{XY},$$

so the two remainder terms are $o(h^{H_{XY}})$. This proves (343). Then Proposition 7.2 gives (344). ☐

Corollary 32.10 (Quantitative observable screening from a quantitative near-layer profile)**.** *Assume the hypotheses of Theorem 32.8. For $0 < h \le h_{0,\alpha}$, define the near-layer screening profile*

$$\omega_\alpha(h) := \sup\left\{h^{-H_{XY}}\,|\mathcal{F}_{t,h}(v)| : v \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},\ \|v\|_{\mathfrak{G}_t} \le 1\right\}.$$

Suppose moreover that there exist $\rho > 0$, $C_{\omega,\alpha} < \infty$, and $\widetilde{h}_{0,\alpha} \in (0, h_{0,\alpha}]$ such that

$$\omega_\alpha(h) \le C_{\omega,\alpha} h^\rho \qquad \text{for every } 0 < h \le \widetilde{h}_{0,\alpha}.$$

Set

$$\varepsilon := \min\{\rho,\ 1-\alpha(1-H_{XY}) - H_{XY},\ H_Y - H_{XY}\}.$$

If $\varepsilon > 0$, then

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = O(h^{H_{XY}+\varepsilon}) \qquad \text{as } h \downarrow 0, \tag{345}$$

and consequently

$$\widetilde{G}_{X\to Y}(t,h) = O(h^{2H_{XY}+2\varepsilon}) \qquad \text{as } h \downarrow 0. \tag{346}$$

Proof. Repeat the proof of Theorem 32.8, but retain the explicit scale. For $\|v\|_{\mathfrak{G}_t} \le 1$, the near-layer term satisfies

$$|\mathcal{F}_{t,h}(w_h)| \le C_{\mathrm{scut},\alpha}\omega_\alpha(h)h^{H_{XY}} \le C_{\mathrm{scut},\alpha}C_{\omega,\alpha}h^{H_{XY}+\rho}.$$

The same far/strip calculation used in Theorem 32.8 gives the unchanged remainder estimate

$$|\mathcal{F}_{t,h}(r_h)| = O\Big(h^{1-\alpha(1-H_{XY})}\Big) + O(h^{H_Y}).$$

Therefore

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} \le Ch^{H_{XY}+\rho} + Ch^{1-\alpha(1-H_{XY})} + Ch^{H_Y}$$

for some finite constant C, uniformly for $0 < h \le \widetilde{h}_{0,\alpha}$. Factoring out $h^{H_{XY}}$ and using the definition of ε gives (345). Proposition 7.2 then yields (346). ☐

33. Reduction of smooth graph localization to exact micro correction

The last reduction is intrinsic to the micro analysis. A smooth time-side cutoff near t becomes a slowly varying cutoff on the moving micro window $[0,\varepsilon_{h,\alpha}^{-1}]$. Observable graph localization is reduced to producing one exact corrected micro output on that window with the required weighted boundary-layer control.

Proposition 33.1 (Exact full h-scale trace identity for a graph element)**.** *Assume Assumptions 3.1 and 4.1, let $\eta>0$, fix $t\in(0,T)$, and suppose*

$$0<H_{XY}<H_Y<\tfrac{1}{2}.$$

Set

$$\beta:=H_Y-H_{XY}.$$

For $i\in\{Y,XY\}$, define the full h-scale Volterra operator

$$\mathfrak{V}^{\mathrm{full}}_{i,t,h}:L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$$

by

$$(\mathfrak{V}^{\mathrm{full}}_{i,t,h}\Theta)(r):=\int_r^{t/h}(q-r)^{H_i-\frac{1}{2}}L_i(t-hr,t-hq)\Theta(q)\,dq,\qquad r\ge 0. \tag{347}$$

Then:

(i) for every $f\in H_t$,

$$\mathcal{D}_{t,h}\mathcal{K}_{i,t}f=h^{H_i}\mathfrak{V}^{\mathrm{full}}_{i,t,h}(\mathcal{D}_{t,h}f)\qquad \text{in } L^2(\mathbb{R}_+); \tag{348}$$

(ii) for every $v\in\mathfrak{G}_t$, if

$$\Phi_{v,h}:=\mathcal{D}_{t,h}v,\qquad \Psi_{v,h}:=h^{\beta}\mathcal{D}_{t,h}\mathcal{S}_tv, \tag{349}$$

then

$$\mathfrak{V}^{\mathrm{full}}_{Y,t,h}\Psi_{v,h}=\mathfrak{V}^{\mathrm{full}}_{XY,t,h}\Phi_{v,h}\qquad \text{in } L^2(\mathbb{R}_+); \tag{350}$$

(iii) if $\chi\in C^\infty([0,1])$, $0\le\chi\le 1$, and

$$\chi\equiv 1 \text{ on } [0,1-\delta_*]\qquad \text{for some } \delta_*\in(0,1),$$

then with $\widetilde{\chi}_{h,\alpha}$ from Proposition 12.12,

$$\mathcal{L}_{t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h})=\chi^{\mathrm{bdry}}_{h,\alpha}v,\qquad \chi^{\mathrm{bdry}}_{h,\alpha}(s):=\chi\Big(\frac{t-s}{h^\alpha}\Big)\mathbf{1}_{[t-h^\alpha,t]}(s). \tag{351}$$

Proof. Fix $i\in\{Y,XY\}$ and $f\in H_t$. For $0\le r\le t/h$,

$$(\mathcal{D}_{t,h}\mathcal{K}_{i,t}f)(r)=h^{1/2}\int_0^{t-hr}(t-hr-s)^{H_i-\frac{1}{2}}L_i(t-hr,s)f(s)\,ds.$$

With the change of variables $s=t-hq$, $ds=-h\,dq$, this becomes

$$\begin{aligned}(\mathcal{D}_{t,h}\mathcal{K}_{i,t}f)(r)&=h^{H_i}\int_r^{t/h}(q-r)^{H_i-\frac{1}{2}}L_i(t-hr,t-hq)\,h^{1/2}f(t-hq)\,dq\\&=h^{H_i}\big(\mathfrak{V}^{\mathrm{full}}_{i,t,h}(\mathcal{D}_{t,h}f)\big)(r).\end{aligned}$$

For $r>t/h$, both sides vanish. This proves (348).

Now let $v\in\mathfrak{G}_t$. Since $\mathcal{K}_{Y,t}\mathcal{S}_tv=\mathcal{K}_{XY,t}v$, applying (348) with $f=\mathcal{S}_tv$ and $f=v$ gives

$$h^{H_Y}\mathfrak{V}^{\mathrm{full}}_{Y,t,h}(\mathcal{D}_{t,h}\mathcal{S}_tv)=h^{H_{XY}}\mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\mathcal{D}_{t,h}v).$$

Since $\beta=H_Y-H_{XY}$, this is (350).

Finally, for $0\le s\le t$,

$$\begin{aligned}(\mathcal{L}_{t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}))(s)&=h^{-1/2}\widetilde{\chi}_{h,\alpha}\Big(\frac{t-s}{h}\Big)\Phi_{v,h}\Big(\frac{t-s}{h}\Big)\mathbf{1}_{[0,t]}(s)\\&=\widetilde{\chi}_{h,\alpha}\Big(\frac{t-s}{h}\Big)v(s)\mathbf{1}_{[0,t]}(s).\end{aligned}$$

By Proposition 12.12,

$$\widetilde{\chi}_{h,\alpha}\Big(\frac{t-s}{h}\Big) = \chi\Big(\varepsilon_{h,\alpha}\frac{t-s}{h}\Big)\,\mathbf{1}_{\left\{0\le\frac{t-s}{h}\le\varepsilon_{h,\alpha}^{-1}\right\}} = \chi\Big(\frac{t-s}{h^\alpha}\Big)\,\mathbf{1}_{[t-h^\alpha,t]}(s).$$

This proves (351). □

Theorem 33.2 (Scaled exact micro-correction criterion for boundary localization)**.** *Assume Assumptions 3.1 and 4.1, let $\eta>0$, fix $t\in(0,T)$, and suppose*

$$0<H_{XY}<H_Y<\tfrac12.$$

Fix $\alpha\in(0,1)$, choose $\chi\in C^\infty([0,1])$ with

$$0\le\chi\le1,\qquad \chi\equiv1 \text{ on } [0,1-\delta_*]$$

for some $\delta_*\in(0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the corresponding moving micro cutoff from Proposition 12.12. Assume in addition that there exist

$$C_{\mathrm{mc},\alpha}<\infty,\qquad h_{0,\alpha}\in(0,t]$$

such that for every $0<h\le h_{0,\alpha}$ and every $v\in\mathfrak{G}_t$, setting

$$\Phi_{v,h}:=\mathcal{D}_{t,h}v,$$

there exists

$$\widetilde{\Psi}_{v,h}\in L^2(\mathbb{R}_+)$$

with

$$\operatorname{supp}(\widetilde{\Psi}_{v,h})\subset[0,\varepsilon_{h,\alpha}^{-1}],\qquad \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\widetilde{\Psi}_{v,h}=\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\big(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}\big) \tag{352}$$

and

$$\|\widetilde{\chi}_{h,\alpha}\Phi_{v,h}\|^2_{L^2(\mathbb{R}_+)}+\frac{\eta^2}{\nu_Y^2}\Big\|\widetilde{\Psi}_{v,h}\Big\|^2_{L^2(\mathbb{R}_+)}\le C^2_{\mathrm{mc},\alpha}\|v\|^2_{\mathfrak{G}_t}. \tag{353}$$

Then, with the time-side cutoff

$$\chi_{h,\alpha}:=\chi^{\mathrm{bdry}}_{h,\alpha}$$

from (351), one has for every $0<h\le h_{0,\alpha}$ and every $v\in\mathfrak{G}_t$,

$$\chi_{h,\alpha}v\in\mathfrak{G}_t\cap H^{\mathrm{near}}_{t,h,\alpha} \tag{354}$$

and

$$\|\chi_{h,\alpha}v\|^2_{H_t}+\frac{\eta^2}{\nu_Y^2}h^{2\beta}\|\mathcal{S}_t(\chi_{h,\alpha}v)\|^2_{H_t}\le C^2_{\mathrm{mc},\alpha}\|v\|^2_{\mathfrak{G}_t}. \tag{355}$$

Proof. Fix $0<h\le h_{0,\alpha}$ and $v\in\mathfrak{G}_t$, and define

$$\widetilde{\Phi}_{v,h}:=\widetilde{\chi}_{h,\alpha}\Phi_{v,h}.$$

Set

$$\phi_{v,h}:=\mathfrak{l}_{\varepsilon_{h,\alpha}}\widetilde{\Phi}_{v,h}\in E,\qquad \psi_{v,h}:=\varepsilon_{h,\alpha}^{-\beta}\mathfrak{l}_{\varepsilon_{h,\alpha}}\widetilde{\Psi}_{v,h}\in E.$$

Applying Proposition 14.5 to (352), and using $\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{d}_{\varepsilon_{h,\alpha}}=I_E$ from Lemma 14.2, gives

$$\mathfrak{A}_{Y,t,h,\alpha}\psi_{v,h}=\mathfrak{A}_{XY,t,h,\alpha}\phi_{v,h}.$$

Hence

$$(\phi_{v,h},\psi_{v,h})\in\Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}.$$

By Proposition 12.8,

$$w_{v,h}:=\mathcal{L}^{(\alpha)}_{t,h}\phi_{v,h}\in\mathfrak{G}_t\cap H^{\mathrm{near}}_{t,h,\alpha}$$

Because $\widetilde{\Phi}_{v,h}$ and $\widetilde{\Psi}_{v,h}$ are supported in $[0,\varepsilon_{h,\alpha}^{-1}]$, Proposition 14.3(ii) gives

$$\mathcal{L}^{(\alpha)}_{t,h}\phi_{v,h}=\mathcal{L}_{t,h}\widetilde{\Phi}_{v,h},$$

so Proposition 33.1 yields

$$w_{v,h}=\mathcal{L}_{t,h}\widetilde{\Phi}_{v,h}=\chi^{\mathrm{bdry}}_{h,\alpha}v.$$

By Proposition 12.8,

$$\mathcal{S}_t w_{v,h} = h^{-\alpha\beta}\mathcal{L}^{(\alpha)}_{t,h}\psi_{v,h} = h^{-\beta}\mathcal{L}_{t,h}\widetilde{\Psi}_{v,h},$$

where we also used $\varepsilon_{h,\alpha} = h^{1-\alpha}$. Lemma 7.4 therefore gives

$$\|w_{v,h}\|^2_{H_t} + \frac{\eta^2}{\nu_Y^2}h^{2\beta}\|\mathcal{S}_t w_{v,h}\|^2_{H_t} = \left\|\widetilde{\Phi}_{v,h}\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\eta^2}{\nu_Y^2}\left\|\widetilde{\Psi}_{v,h}\right\|^2_{L^2(\mathbb{R}_+)}. \tag{356}$$

Combining (353) with (356) yields

$$\chi^{\mathrm{bdry}}_{h,\alpha} v \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha},$$

and

$$\|\chi^{\mathrm{bdry}}_{h,\alpha} v\|^2_{H_t} + \frac{\eta^2}{\nu_Y^2}h^{2\beta}\|\mathcal{S}_t(\chi^{\mathrm{bdry}}_{h,\alpha} v)\|^2_{H_t} \le C^2_{\mathrm{mc},\alpha}\|v\|^2_{\mathfrak{G}_t}.$$

By construction,

$$\chi^{\mathrm{bdry}}_{h,\alpha}(s) = \chi\left(\frac{t-s}{h^\alpha}\right)\mathbf{1}_{[t-h^\alpha,t]}(s),$$

so

$$0 \le \chi^{\mathrm{bdry}}_{h,\alpha} \le 1, \qquad \chi^{\mathrm{bdry}}_{h,\alpha} \equiv 1 \text{ on } [t-(1-\delta_*)h^\alpha, t], \qquad \operatorname{supp}\chi^{\mathrm{bdry}}_{h,\alpha} \subset [t-h^\alpha, t].$$

This proves (354) and (355). □

Remark 33.3 (Exact micro correction and smooth localization). Theorem 33.2 supplies the scaled boundary-localization mechanism used in the closing reduction. It already gives the correct time-side cutoff in the boundary-layer norm, and isolates the only missing upgrade to Definition 32.3: the factor h^β in front of $\|\mathcal{S}_t(\chi_{h,\alpha}v)\|_{H_t}$. By Proposition 33.1, the full-h trace pair is exact before cutoff, so that upgrade is encoded entirely by the moving exact-cutoff defect. The later sections resolve it by passing to the growing-window exact defect and comparing that object with its frozen commutator model.

Definition 33.4 (Moving exact-cutoff defect for a graph element)**.** Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, let $\alpha \in (0,1)$, and suppose

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Set

$$\beta := H_Y - H_{XY}.$$

Let $\chi \in C^\infty([0,1])$ satisfy

$$0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff from Proposition 12.12. For $v \in \mathfrak{G}_t$ and $h > 0$, define the exact full-h traces

$$\Phi_{v,h} := \mathcal{D}_{t,h}v, \qquad \Psi_{v,h} := h^\beta \mathcal{D}_{t,h}\mathcal{S}_t v,$$

and the corresponding moving exact-cutoff defect by

$$\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha} := \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}) - \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}(\widetilde{\chi}_{h,\alpha}\Psi_{v,h}). \tag{357}$$

Proposition 33.5 (Exact commutator-tail decomposition of the moving exact-cutoff defect)**.** *Assume the hypotheses of Definition 33.4. Let $v \in \mathfrak{G}_t$, let $0 < h \le t$ satisfy $h^\alpha \le t$, and set*

$$\varepsilon := \varepsilon_{h,\alpha} = h^{1-\alpha}.$$

With $\Phi_{v,h}$, $\Psi_{v,h}$, and $\widetilde{\chi}_{h,\alpha}$ as above, define

$$(\mathfrak{C}^{\mathrm{cut}}_{XY,v,h,\alpha})(r) := \int_r^{\varepsilon^{-1}} (q-r)^{H_{XY}-\frac{1}{2}}\big(\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)\big)L_{XY}(t-hr, t-hq)\Phi_{v,h}(q)\,dq, \tag{358}$$

$$(\mathfrak{C}^{\mathrm{cut}}_{Y,v,h,\alpha})(r) := \int_r^{\varepsilon^{-1}} (q-r)^{H_{Y}-\frac{1}{2}}\big(\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)\big)L_{Y}(t-hr, t-hq)\Psi_{v,h}(q)\,dq, \tag{359}$$

for $r \geq 0$, and define the old-tail forcing by

$$(\mathfrak{O}^{\mathrm{old}}_{v,h,\alpha})(r) := \mathbf{1}_{[0,\varepsilon^{-1}]}(r) \int_{\varepsilon^{-1}}^{t/h} \Big[(q-r)^{H_Y - \frac{1}{2}} L_Y(t-hr, t-hq)\Psi_{v,h}(q) - (q-r)^{H_{XY}-\frac{1}{2}} L_{XY}(t-hr,t-hq)\Phi_{v,h}(q)\Big]\, dq. \tag{360}$$

Then:

(i) one has the exact decomposition

$$\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha} = \mathfrak{C}^{\mathrm{cut}}_{XY,v,h,\alpha} - \mathfrak{C}^{\mathrm{cut}}_{Y,v,h,\alpha} + \widetilde{\chi}_{h,\alpha}\mathfrak{O}^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } L^2(\mathbb{R}_+); \tag{361}$$

(ii) defining the unit-interval traces

$$\phi^{\sharp}_{v,h} := \mathfrak{l}_\varepsilon \Phi_{v,h} \in E, \qquad \psi^{\sharp}_{v,h} := \varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi_{v,h} \in E, \tag{362}$$

and the rescaled old-tail source

$$\omega^{\mathrm{old}}_{v,h,\alpha} := \varepsilon^{H_{XY}} \mathfrak{l}_\varepsilon \mathfrak{O}^{\mathrm{old}}_{v,h,\alpha} \in E, \tag{363}$$

one has

$$\mathfrak{A}_{XY,t,h,\alpha}\phi^{\sharp}_{v,h} - \mathfrak{A}_{Y,t,h,\alpha}\psi^{\sharp}_{v,h} = \omega^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } E; \tag{364}$$

(iii) with the same cutoff χ on $[0,1]$ used to define $\widetilde{\chi}_{h,\alpha}$,

$$\varepsilon^{H_{XY}}\mathfrak{l}_\varepsilon \mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha} = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^{\sharp}_{v,h} - \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} + \chi\,\omega^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } E. \tag{365}$$

Proof. For $r > \varepsilon^{-1}$, all terms in (361) vanish, so it is enough to work on $0 \leq r \leq \varepsilon^{-1}$. There,

$$\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha}(r) = \int_r^{\varepsilon^{-1}} (q-r)^{H_{XY}-\frac{1}{2}} L_{XY}(t-hr,t-hq)\widetilde{\chi}_{h,\alpha}(q)\Phi_{v,h}(q)\,dq - \int_r^{\varepsilon^{-1}} (q-r)^{H_Y-\frac{1}{2}} L_Y(t-hr,t-hq)\widetilde{\chi}_{h,\alpha}(q)\Psi_{v,h}(q)\,dq.$$

Adding and subtracting $\widetilde{\chi}_{h,\alpha}(r)$ inside both integrals gives

$$\begin{aligned}\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha}(r) &= \mathfrak{C}^{\mathrm{cut}}_{XY,v,h,\alpha}(r) - \mathfrak{C}^{\mathrm{cut}}_{Y,v,h,\alpha}(r) \\ &\quad + \widetilde{\chi}_{h,\alpha}(r)\Bigg[\int_r^{\varepsilon^{-1}} (q-r)^{H_{XY}-\frac{1}{2}} L_{XY}(t-hr,t-hq)\Phi_{v,h}(q)\,dq \\ &\qquad - \int_r^{\varepsilon^{-1}} (q-r)^{H_Y-\frac{1}{2}} L_Y(t-hr,t-hq)\Psi_{v,h}(q)\,dq\Bigg].\end{aligned}$$

By Proposition 33.1(ii),

$$\int_r^{t/h} (q-r)^{H_{XY}-\frac{1}{2}} L_{XY}(t-hr,t-hq)\Phi_{v,h}(q)\,dq = \int_r^{t/h} (q-r)^{H_Y-\frac{1}{2}} L_Y(t-hr,t-hq)\Psi_{v,h}(q)\,dq.$$

Subtracting the common tail from $q = \varepsilon^{-1}$ to $q = t/h$ yields

$$\begin{aligned}&\int_r^{\varepsilon^{-1}} (q-r)^{H_{XY}-\frac{1}{2}} \\ &\qquad \times L_{XY}(t-hr,t-hq)\Phi_{v,h}(q)\,dq \\ &\quad - \int_r^{\varepsilon^{-1}} (q-r)^{H_Y-\frac{1}{2}} \\ &\qquad \times L_Y(t-hr,t-hq)\Psi_{v,h}(q)\,dq \\ &= \mathfrak{O}^{\mathrm{old}}_{v,h,\alpha}(r),\end{aligned}$$

which proves (361).

For item (ii), Proposition 14.5 gives

$$\mathfrak{d}_\varepsilon \mathfrak{A}_{XY,t,h,\alpha}\phi^{\sharp}_{v,h} = \varepsilon^{H_{XY}}\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi_{v,h},$$

and, since $\beta = H_Y - H_{XY}$,

$$\mathfrak{d}_\varepsilon \mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h} = \varepsilon^{H_Y-\beta}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_{v,h} = \varepsilon^{H_{XY}}\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_{v,h}.$$

Therefore

$$\mathfrak{d}_\varepsilon\big(\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h} - \mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}\big) = \varepsilon^{H_{XY}}\big(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\Phi_{v,h} - \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\Psi_{v,h}\big).$$

By the full-tail identity established in the proof of item (i), the bracket on the right is exactly $\mathfrak{O}^{\mathrm{old}}_{v,h,\alpha}$. Applying $\mathfrak{l}_\varepsilon$ and using $\mathfrak{l}_\varepsilon\mathfrak{d}_\varepsilon = I_E$ from Lemma 14.2 proves (364).

For item (iii), Proposition 12.12 gives

$$\mathfrak{l}_\varepsilon\big(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}\big) = \chi\,\phi^\sharp_{v,h}, \qquad \mathfrak{l}_\varepsilon\big(\widetilde{\chi}_{h,\alpha}\Psi_{v,h}\big) = \varepsilon^\beta\chi\,\psi^\sharp_{v,h}.$$

Applying Proposition 14.5 to these two identities yields

$$\varepsilon^{H_{XY}}\mathfrak{l}_\varepsilon\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha} = \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}).$$

Now expand the right-hand side by adding and subtracting $\chi\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h}$ and $\chi\mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}$:

$$\begin{aligned}\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}) &= \big(\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \chi\,\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h}\big)\\ &\quad - \big(\mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}) - \chi\,\mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}\big)\\ &\quad + \chi\big(\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h} - \mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}\big).\end{aligned}$$

By the definitions (139)–(140), the first two brackets equal

$$\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h}, \qquad \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h},$$

respectively. Using (364) for the last term gives (365). □

Definition 33.6 (Unit-interval source for the moving cutoff problem)**.** Under the hypotheses and notation of Proposition 33.5, define the unit-interval source term by

$$\Sigma^{\mathrm{cut}}_{v,h,\alpha} := \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} - \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h} + \chi\,\omega^{\mathrm{old}}_{v,h,\alpha} \in E. \tag{366}$$

Theorem 33.7 (Unscaled exact-correction criterion for boundary graph localization)**.** *Assume Assumptions 3.1 and 4.1, let $\eta > 0$, fix $t \in (0,T)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Fix $\alpha \in (0,1)$, choose $\chi \in C^\infty([0,1])$ with

$$0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff from Proposition 12.12. Assume that there exist

$$C_{\mathrm{tail},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$, with $\Phi_{v,h}$, $\Psi_{v,h}$, and $\mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha}$ as in Definition 33.4, there exists

$$\mathfrak{C}_{v,h} \in L^2(\mathbb{R}_+)$$

satisfying

$$\operatorname{supp}(\mathfrak{C}_{v,h}) \subset [0, \varepsilon^{-1}_{h,\alpha}], \qquad \mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\mathfrak{C}_{v,h} = \mathfrak{R}^{\mathrm{cut}}_{v,h,\alpha} \tag{367}$$

and

$$\|\mathfrak{C}_{v,h}\|_{L^2(\mathbb{R}_+)} \le C_{\mathrm{tail},\alpha}h^\beta\|v\|_{\mathfrak{G}_t}. \tag{368}$$

Then $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t in the sense of Definition 32.3. The cutoff can be chosen with constants

$$C_{\mathrm{scut},\alpha} < \infty, \qquad h_{1,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{1,\alpha}$, the cutoff

$$\chi_{h,\alpha}(s) := \chi\Big(\frac{t-s}{h^\alpha}\Big)\,\mathbf{1}_{[t-h^\alpha,t]}(s)$$

satisfies

$$0 \le \chi_{h,\alpha} \le 1, \qquad \chi_{h,\alpha} \equiv 1 \text{ on } [t-(1-\delta_*)h^\alpha, t], \qquad \operatorname{supp}\chi_{h,\alpha} \subset [t-h^\alpha, t],$$

and for every $v \in \mathfrak{G}_t$,

$$\chi_{h,\alpha} v \in \mathfrak{G}_t \qquad \text{and} \qquad \|\chi_{h,\alpha} v\|_{\mathfrak{G}_t} \le C_{\mathrm{scut},\alpha}\|v\|_{\mathfrak{G}_t}. \tag{369}$$

Proof. Set

$$\beta := H_Y - H_{XY}.$$

Fix $0 < h \le h_{0,\alpha}$ and $v \in \mathfrak{G}_t$, and abbreviate

$$\widetilde{\Phi}_{v,h} := \widetilde{\chi}_{h,\alpha}\Phi_{v,h}, \qquad \widetilde{\Psi}_{v,h} := \widetilde{\chi}_{h,\alpha}\Psi_{v,h} + \mathfrak{C}_{v,h}.$$

By (367) and Definition 33.4,

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha}\widetilde{\Psi}_{v,h} = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha}\widetilde{\Phi}_{v,h}.$$

Moreover,

$$\operatorname{supp}(\widetilde{\Phi}_{v,h}) \subset [0, \varepsilon_{h,\alpha}^{-1}], \qquad \operatorname{supp}(\widetilde{\Psi}_{v,h}) \subset [0, \varepsilon_{h,\alpha}^{-1}],$$

because $\widetilde{\chi}_{h,\alpha}$ and $\mathfrak{C}_{v,h}$ are supported in $[0, \varepsilon_{h,\alpha}^{-1}]$.

By Lemma 7.4,

$$\|\widetilde{\Phi}_{v,h}\|_{L^2(\mathbb{R}_+)} \le \|\Phi_{v,h}\|_{L^2(\mathbb{R}_+)} = \|v\|_{H_t} \le \|v\|_{\mathfrak{G}_t}.$$

Likewise,

$$\|\widetilde{\chi}_{h,\alpha}\Psi_{v,h}\|_{L^2(\mathbb{R}_+)} \le \|\Psi_{v,h}\|_{L^2(\mathbb{R}_+)} = h^\beta\|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta}h^\beta\|v\|_{\mathfrak{G}_t}.$$

Combining this with (368) yields

$$\|\widetilde{\Psi}_{v,h}\|_{L^2(\mathbb{R}_+)} \le \left(\frac{\nu_Y}{\eta} + C_{\mathrm{tail},\alpha}\right)h^\beta\|v\|_{\mathfrak{G}_t}. \tag{370}$$

Hence there exists $C_{\mathrm{mc},\alpha} < \infty$ such that

$$\|\widetilde{\Phi}_{v,h}\|^2_{L^2(\mathbb{R}_+)} + \frac{\eta^2}{\nu_Y^2}h^{-2\beta}\|\widetilde{\Psi}_{v,h}\|^2_{L^2(\mathbb{R}_+)} \le C^2_{\mathrm{mc},\alpha}\|v\|^2_{\mathfrak{G}_t}. \tag{371}$$

Set

$$\phi_{v,h} := \mathfrak{l}_{\varepsilon_{h,\alpha}}\widetilde{\Phi}_{v,h} \in E, \qquad \psi_{v,h} := \varepsilon_{h,\alpha}^{-\beta}\mathfrak{l}_{\varepsilon_{h,\alpha}}\widetilde{\Psi}_{v,h} \in E.$$

Applying Proposition 14.5 to the exact micro identity above, and using $\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{d}_{\varepsilon_{h,\alpha}} = I_E$ from Lemma 14.2, gives

$$\mathfrak{A}_{Y,t,h,\alpha}\psi_{v,h} = \mathfrak{A}_{XY,t,h,\alpha}\phi_{v,h}.$$

So

$$(\phi_{v,h}, \psi_{v,h}) \in \Gamma^{\mathrm{loc,ex}}_{t,h,\alpha}.$$

By Proposition 12.8,

$$w_{v,h} := \mathcal{L}^{(\alpha)}_{t,h}\phi_{v,h} \in \mathfrak{G}_t \cap H^{\mathrm{near}}_{t,h,\alpha}$$

and

$$\mathcal{S}_t w_{v,h} = h^{-\alpha\beta}\mathcal{L}^{(\alpha)}_{t,h}\psi_{v,h}.$$

Because $\widetilde{\Phi}_{v,h}$ and $\widetilde{\Psi}_{v,h}$ are supported in $[0, \varepsilon_{h,\alpha}^{-1}]$, Proposition 14.3(ii) gives

$$\mathcal{L}^{(\alpha)}_{t,h}\phi_{v,h} = \mathcal{L}_{t,h}\widetilde{\Phi}_{v,h}, \qquad \mathcal{L}^{(\alpha)}_{t,h}\psi_{v,h} = \varepsilon_{h,\alpha}^{-\beta}\mathcal{L}_{t,h}\widetilde{\Psi}_{v,h}.$$

Since $\varepsilon_{h,\alpha} = h^{1-\alpha}$, it follows that

$$\mathcal{S}_t w_{v,h} = h^{-\beta}\mathcal{L}_{t,h}\widetilde{\Psi}_{v,h}.$$

Also, Proposition 33.1 gives

$$w_{v,h} = \mathcal{L}_{t,h}\widetilde{\Phi}_{v,h} = \chi_{h,\alpha}v.$$

Therefore $\chi_{h,\alpha}v \in \mathfrak{G}_t$. Using Lemma 7.4,

$$\begin{aligned}\|\chi_{h,\alpha}v\|_{\mathfrak{G}_t}^2 &= \|w_{v,h}\|_{H_t}^2 + \frac{\eta^2}{\nu_Y^2}\|\mathcal{S}_t w_{v,h}\|_{H_t}^2 \\ &= \|\widetilde{\Phi}_{v,h}\|_{L^2(\mathbb{R}_+)}^2 + \frac{\eta^2}{\nu_Y^2}h^{-2\beta}\|\widetilde{\Psi}_{v,h}\|_{L^2(\mathbb{R}_+)}^2 \\ &\le C_{\mathrm{mc},\alpha}^2\|v\|_{\mathfrak{G}_t}^2\end{aligned}$$

by (371). This proves (369).

Finally, by definition,

$$\chi_{h,\alpha}(s) = \chi\left(\frac{t-s}{h^\alpha}\right)\mathbf{1}_{[t-h^\alpha,t]}(s),$$

so the stated support and plateau properties hold. Thus Definition 32.3 is satisfied with $h_{1,\alpha} := h_{0,\alpha}$ and $C_{\mathrm{scut},\alpha} := C_{\mathrm{mc},\alpha}$. $\square$

Corollary 33.8 (Unit-interval sufficient criterion for boundary graph localization)**.** *Assume the hypotheses of Proposition 33.5. Fix*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Assume that there exist

$$C_{\mathrm{loc},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$, with $\Sigma_{v,h,\alpha}^{\mathrm{cut}}$ from Definition 33.6, there exists

$$\zeta_{v,h}^{\mathrm{cut}} \in E$$

such that

$$\mathfrak{A}_{Y,t,h,\alpha}\zeta_{v,h}^{\mathrm{cut}} = \Sigma_{v,h,\alpha}^{\mathrm{cut}} \qquad \text{in } E \tag{372}$$

and

$$\|\zeta_{v,h}^{\mathrm{cut}}\|_E \le C_{\mathrm{loc},\alpha}h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}, \qquad \beta := H_Y - H_{XY}. \tag{373}$$

Then the conclusion of Theorem 33.7 holds. In particular, $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and the screening conclusions (466)–(467) *hold.*

Proof. Fix $0 < h \le h_{0,\alpha}$ and $v \in \mathfrak{G}_t$, and set

$$\varepsilon := \varepsilon_{h,\alpha} = h^{1-\alpha}.$$

Define the micro correction

$$\mathfrak{C}_{v,h} := \varepsilon^\beta\mathfrak{d}_\varepsilon\zeta_{v,h}^{\mathrm{cut}} \in L^2(\mathbb{R}_+). \tag{374}$$

By Lemma 14.2,

$$\operatorname{supp}(\mathfrak{C}_{v,h}) \subset [0, \varepsilon^{-1}]$$

and

$$\|\mathfrak{C}_{v,h}\|_{L^2(\mathbb{R}_+)} = \varepsilon^\beta\|\zeta_{v,h}^{\mathrm{cut}}\|_E \le \varepsilon^\beta C_{\mathrm{loc},\alpha}h^{\alpha\beta}\|v\|_{\mathfrak{G}_t} = C_{\mathrm{loc},\alpha}h^\beta\|v\|_{\mathfrak{G}_t}.$$

By Proposition 14.5,

$$\begin{aligned}\mathfrak{V}_{Y,t,h,\alpha}^{\mathrm{mic}}\mathfrak{C}_{v,h} &= \varepsilon^{-H_Y}\mathfrak{d}_\varepsilon\mathfrak{A}_{Y,t,h,\alpha}\mathfrak{l}_\varepsilon(\varepsilon^\beta\mathfrak{d}_\varepsilon\zeta_{v,h}^{\mathrm{cut}}) \\ &= \varepsilon^{-H_Y+\beta}\mathfrak{d}_\varepsilon\mathfrak{A}_{Y,t,h,\alpha}\zeta_{v,h}^{\mathrm{cut}} \\ &= \varepsilon^{-H_{XY}}\mathfrak{d}_\varepsilon\Sigma_{v,h,\alpha}^{\mathrm{cut}},\end{aligned}$$

where we used $\mathfrak{l}_\varepsilon\mathfrak{d}_\varepsilon = I_E$ and $\beta = H_Y - H_{XY}$. By (365),

$$\varepsilon^{-H_{XY}}\mathfrak{d}_\varepsilon\Sigma_{v,h,\alpha}^{\mathrm{cut}} = \mathfrak{R}_{v,h,\alpha}^{\mathrm{cut}}.$$

Hence

$$\mathfrak{V}_{Y,t,h,\alpha}^{\mathrm{mic}}\mathfrak{C}_{v,h} = \mathfrak{R}_{v,h,\alpha}^{\mathrm{cut}}.$$

So the hypotheses (367)–(368) of Theorem 33.7 are satisfied. The conclusion now follows from that theorem together with Corollary 33.53. $\square$

Corollary 33.9 (Exact fixed-window defect criterion on the unit interval)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Assume that there exist

$$C_{\mathrm{def},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$, with $\phi^\sharp_{v,h}$ and $\psi^\sharp_{v,h}$ from (362)*, one has*

$$\chi\phi^\sharp_{v,h} \in H^\beta((0,1)), \qquad \beta := H_Y - H_{XY},$$

and

$$\left\| \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}) \right\|_E \le C_{\mathrm{def},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}. \tag{375}$$

Then the hypotheses of Corollary 33.8 are satisfied. In particular, $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and the screening conclusions (466)– (467) *hold.*

Proof. Fix $0 < h \le h_{0,\alpha}$ and $v \in \mathfrak{G}_t$. Since

$$\chi\phi^\sharp_{v,h} \in H^\beta((0,1)), \qquad \chi\psi^\sharp_{v,h} \in E,$$

Theorem 19.6 with $R = 1$ applies to the pair

$$(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}).$$

Define

$$\widetilde{\psi}^\sharp_{v,h} := \mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^\sharp_{v,h}) \in E.$$

Then

$$(\chi\phi^\sharp_{v,h}, \widetilde{\psi}^\sharp_{v,h}) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,1},$$

and

$$\left\| \widetilde{\psi}^\sharp_{v,h} - \chi\psi^\sharp_{v,h} \right\|_E \le C_{t,1} \left\| \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}) \right\|_E, \tag{376}$$

by (227). Set

$$\xi^{\mathrm{cut}}_{v,h} := \widetilde{\psi}^\sharp_{v,h} - \chi\psi^\sharp_{v,h} \in E.$$

Using (375) in (376) yields

$$\|\xi^{\mathrm{cut}}_{v,h}\|_E \le C_{t,1} C_{\mathrm{def},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}.$$

Because $(\chi\phi^\sharp_{v,h}, \widetilde{\psi}^\sharp_{v,h}) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,1}$, one has

$$\mathfrak{A}_{Y,t,h,\alpha}\widetilde{\psi}^\sharp_{v,h} = \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) \qquad \text{in } E.$$

Therefore

$$\mathfrak{A}_{Y,t,h,\alpha}\xi^{\mathrm{cut}}_{v,h} = \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}).$$

Now add and subtract $\chi\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h}$ and $\chi\mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}$, and use (364). This gives

$$\begin{aligned}
\mathfrak{A}_{Y,t,h,\alpha}\xi^{\mathrm{cut}}_{v,h} &= \left(\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \chi\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h}\right) \\
&\quad - \left(\mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}) - \chi\mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}\right) \\
&\quad + \chi\left(\mathfrak{A}_{XY,t,h,\alpha}\phi^\sharp_{v,h} - \mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}\right) \\
&= \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} - \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h} + \chi\,\omega^{\mathrm{old}}_{v,h,\alpha} \\
&= \Sigma^{\mathrm{cut}}_{v,h,\alpha} \qquad \text{in } E,
\end{aligned}$$

where the last identity is Definition 33.6. Thus (372) and (373) hold with

$$C_{\mathrm{loc},\alpha} := C_{t,1} C_{\mathrm{def},\alpha}.$$

Corollary 33.8 now gives the conclusion. □

Corollary 33.10 (Source-only fixed-window criterion on the unit interval)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$*, and let* $\widetilde{\chi}_{h,\alpha}$ *be the associated moving micro cutoff. Let*

$$\beta := H_Y - H_{XY}.$$

Assume that there exist

$$C_{\mathrm{src},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ *and every* $v \in \mathfrak{G}_t$*, with* $\phi^\sharp_{v,h}$ *and* $\psi^\sharp_{v,h}$ *from* (362)*, the distribution*

$$\mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} := -\frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)} \mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1} \mathfrak{D}^{\mathrm{dist}}_{\beta,1}(\chi\phi^\sharp_{v,h}) - \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi^\sharp_{v,h}) \tag{377}$$

belongs to $E = L^2([0,1])$ *and satisfies*

$$\left\| \mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} \right\|_E \le C_{\mathrm{src},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}. \tag{378}$$

Then the hypotheses of Corollary 33.9 are satisfied. In particular, $\mathcal{S}_t$ *has* h^α*-stable boundary cutoffs at time* t*, and the screening conclusions* (466)– (467) *hold.*

Proof. After shrinking $h_{0,\alpha}$ if needed, we may assume that Theorem 19.4 and Proposition 15.3 both apply with $R = 1$ for every $0 < h \le h_{0,\alpha}$.

Fix $0 < h \le h_{0,\alpha}$ and $v \in \mathfrak{G}_t$, and set

$$\Phi^\chi_{v,h} := \chi\phi^\sharp_{v,h} \in E.$$

By (377),

$$\frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)} \mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1} \mathfrak{D}^{\mathrm{dist}}_{\beta,1} \Phi^\chi_{v,h} = -\mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} - \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1} \Phi^\chi_{v,h} \qquad \text{in } \mathcal{D}'((0,1)).$$

The right-hand side belongs to E, because $\mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} \in E$ by assumption and $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1} \in \mathcal{L}(E)$ by Theorem 19.4. Since $\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}$ is boundedly invertible for small h, it follows that

$$\mathfrak{D}^{\mathrm{dist}}_{\beta,1} \Phi^\chi_{v,h} \in E.$$

Lemma 19.2(i) therefore gives

$$\chi\phi^\sharp_{v,h} \in H^\beta((0,1)).$$

Now define

$$\Psi^\chi_{v,h} := \chi\psi^\sharp_{v,h} \in E.$$

By (225), the exact fixed-window defect of the pair $(\Phi^\chi_{v,h}, \Psi^\chi_{v,h})$ is

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\Phi^\chi_{v,h}, \Psi^\chi_{v,h}) = \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1} \Psi^\chi_{v,h} + \mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} \qquad \text{in } E.$$

Therefore

$$\left\| \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}) \right\|_E \le \left\| \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}(\chi\psi^\sharp_{v,h}) \right\|_E + \left\| \mathfrak{S}^{\mathrm{cut}}_{v,h,\alpha} \right\|_E. \tag{379}$$

By Proposition 15.3(ii), there exists $C_t < \infty$ such that

$$\left\| \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1} \right\|_{\mathcal{L}(E)} \le C_t \qquad \text{for } 0 < h \le h_{0,\alpha}.$$

Also,

$$\|\chi\psi^\sharp_{v,h}\|_E \le \|\psi^\sharp_{v,h}\|_E = \varepsilon_{h,\alpha}^{-\beta} \|\Psi_{v,h}\|_{L^2([0,\varepsilon_{h,\alpha}^{-1}])}.$$

Using $\Psi_{v,h} = h^\beta \mathcal{D}_{t,h}\mathcal{S}_t v$, Lemma 7.4, and $\varepsilon_{h,\alpha} = h^{1-\alpha}$, we get

$$\|\psi^\sharp_{v,h}\|_E \le \varepsilon_{h,\alpha}^{-\beta} h^\beta \|\mathcal{S}_t v\|_{H_t} = h^{\alpha\beta} \|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}.$$

Hence

$$\left\| \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}(\chi\psi^\sharp_{v,h}) \right\|_E \le C_t \frac{\nu_Y}{\eta} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}.$$

Combining this with (379) and (378) proves (375). Corollary 33.9 now gives the conclusion. □

Definition 33.11 (Conjugated old-tail source on the unit interval)**.** Under the hypotheses of Proposition 33.5, set

$$\lambda := H_Y + \frac{1}{2}.$$

For $v \in \mathfrak{G}_t$ and $0 < h \le t$, define the conjugated old-tail source by

$$\mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} := \frac{1}{\Gamma(\lambda)} \mathfrak{D}^{\lambda}_{-,1}\big(\chi\, \omega^{\mathrm{old}}_{v,h,\alpha}\big) \in \mathcal{D}'((0,1)), \tag{380}$$

where $\omega^{\mathrm{old}}_{v,h,\alpha}$ is given by (363).

Proposition 33.12 (Exact conjugated decomposition of the unit-interval defect)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Let $v \in \mathfrak{G}_t$, let $0 < h \le t$ satisfy $h^\alpha \le t$, and suppose*

$$\chi\phi^\sharp_{v,h} \in H^\beta((0,1)), \qquad \beta := H_Y - H_{XY}.$$

Then, in $\mathcal{D}'((0,1))$,

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}\big(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}\big) = \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h} - \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} - \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha}. \tag{381}$$

In particular, if

$$\mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} + \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} \in E,$$

then $\mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} \in E$ and (381) *holds in $E = L^2([0,1])$.*

Proof. Because $\chi\phi^\sharp_{v,h} \in H^\beta((0,1))$ and $\chi\psi^\sharp_{v,h} \in E$, the exact fixed-window defect

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}\big(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}\big)$$

is well-defined by (225). By (365),

$$\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h}) - \mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}) = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} - \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h} + \chi\,\omega^{\mathrm{old}}_{v,h,\alpha}$$

in E. Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^\lambda_{-,1}$ in the distributional sense gives

$$\begin{aligned}\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\big(\mathfrak{A}_{Y,t,h,\alpha}(\chi\psi^\sharp_{v,h}) - \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^\sharp_{v,h})\big) &= \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^\sharp_{v,h} \\ &\quad - \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^\sharp_{v,h} - \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha},\end{aligned}$$

where we used Proposition 12.9 and Definition 33.11. On the other hand, Proposition 19.5 with $R = 1$ identifies the left-hand side exactly with

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}\big(\chi\phi^\sharp_{v,h}, \chi\psi^\sharp_{v,h}\big)$$

in $\mathcal{D}'((0,1))$. This proves (381).

If the displayed sum belongs to E, then the right-hand side of (381) is an E-distribution minus $\mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha}$, while the left-hand side belongs to E by (225). Hence $\mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} \in E$, and the identity holds in E. □

Corollary 33.13 (Reduction to source-old-tail cancellation on the unit interval)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Assume that there exist

$$C_{\mathrm{tail},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$, with $\phi^\sharp_{v,h}$, $\psi^\sharp_{v,h}$, and $\mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha}$ as above, one has

$$\chi\phi^\sharp_{v,h} \in H^\beta((0,1)), \qquad \beta := H_Y - H_{XY},$$

and

$$\left\| \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \phi^{\sharp}_{v,h} + \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} \right\|_E \le C_{\mathrm{tail},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}. \tag{382}$$

Then the conclusion of Corollary 33.9 holds. In particular, $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and the screening conclusions (466)– (467) *hold.*

Proof. After shrinking $h_{0,\alpha}$ if needed, Proposition 12.9 applies for every $0 < h \le h_{0,\alpha}$. Fix such an h and $v \in \mathfrak{G}_t$. By Proposition 33.12,

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^{\sharp}_{v,h}, \chi\psi^{\sharp}_{v,h}) = \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \psi^{\sharp}_{v,h} - \left(\mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \phi^{\sharp}_{v,h} + \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha} \right)$$

in E.

By (143), there exists $C_t < \infty$, depending only on t and the fixed cutoff χ, such that

$$\left\| \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \right\|_{\mathcal{L}(E)} \le C_t.$$

Also,

$$\|\psi^{\sharp}_{v,h}\|_E = \varepsilon_{h,\alpha}^{-\beta} \|\Psi_{v,h}\|_{L^2([0,\varepsilon_{h,\alpha}^{-1}])} \le \varepsilon_{h,\alpha}^{-\beta} h^{\beta} \|\mathcal{S}_t v\|_{H_t} = h^{\alpha\beta} \|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t},$$

using $\Psi_{v,h} = h^{\beta} \mathcal{D}_{t,h} \mathcal{S}_t v$, Lemma 7.4, and $\varepsilon_{h,\alpha} = h^{1-\alpha}$. Therefore

$$\left\| \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \psi^{\sharp}_{v,h} \right\|_E \le C_t \frac{\nu_Y}{\eta} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}.$$

Combining this with (382) gives

$$\left\| \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^{\sharp}_{v,h}, \chi\psi^{\sharp}_{v,h}) \right\|_E \le C'_{t,\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t},$$

for some $C'_{t,\alpha} < \infty$. Corollary 33.9 now gives the conclusion. □

Definition 33.14 (Preconjugated source-old-tail discrepancy on the unit interval)**.** Under the hypotheses of Proposition 33.5, define

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} := \mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^{\sharp}_{v,h}) - \chi\, \mathfrak{A}_{Y,t,h,\alpha} \psi^{\sharp}_{v,h} \in E. \tag{383}$$

Proposition 33.15 (Exact source-old-tail recombination on the unit interval)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Let $v \in \mathfrak{G}_t$, let $0 < h \le t$ satisfy $h^\alpha \le t$, and suppose*

$$\chi\phi^{\sharp}_{v,h} \in H^{\beta}((0,1)), \qquad \beta := H_Y - H_{XY}.$$

Then:

(i) one has the exact preconjugated identity

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \phi^{\sharp}_{v,h} + \chi\, \omega^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } E; \tag{384}$$

(ii) in $\mathcal{D}'((0,1))$,

$$\frac{1}{\Gamma(\lambda)} \mathfrak{D}^{\lambda}_{-,1} \mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{Q}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \phi^{\sharp}_{v,h} + \mathfrak{O}^{\mathrm{conj}}_{v,h,\alpha}, \qquad \lambda := H_Y + \frac{1}{2}; \tag{385}$$

(iii) consequently,

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^{\sharp}_{v,h}, \chi\psi^{\sharp}_{v,h}) = \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \psi^{\sharp}_{v,h} - \frac{1}{\Gamma(\lambda)} \mathfrak{D}^{\lambda}_{-,1} \mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} \qquad \text{in } \mathcal{D}'((0,1)). \tag{386}$$

If the right-hand side of (385) *belongs to E, then* (385) *and* (386) *both hold in $E = L^2([0,1])$.*

Proof. By definition of $\mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}$,

$$\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^{\sharp}_{v,h}) = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi} \phi^{\sharp}_{v,h} + \chi\, \mathfrak{A}_{XY,t,h,\alpha} \phi^{\sharp}_{v,h} \qquad \text{in } E.$$

Using (364),

$$\mathfrak{A}_{XY,t,h,\alpha} \phi^{\sharp}_{v,h} - \mathfrak{A}_{Y,t,h,\alpha} \psi^{\sharp}_{v,h} = \omega^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } E.$$

Subtracting $\chi\, \mathfrak{A}_{Y,t,h,\alpha} \psi^{\sharp}_{v,h}$ proves (384).

Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^{\lambda}_{-,1}$ to (384), Proposition 12.9, and Definition 33.11 gives (385).

Finally, substituting (385) into (381) from Proposition 33.12 yields (386). If (385) belongs to E, then the left-hand side of (385) is an E-distribution, so (385) holds in E. Since $\mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} \in E$, the right-hand side of (386) also belongs to E, and therefore (386) holds in E as well. □

Proposition 33.16 (Automatic H^{β}-regularity from the preconjugated source discrepancy)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^{\infty}([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Then there exist $h_{0,\alpha} \in (0,t]$ and $C_{t,\alpha} < \infty$ such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$, if

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} \in E, \qquad \lambda := H_Y + \frac{1}{2}, \tag{387}$$

then

$$\chi\phi^{\sharp}_{v,h} \in H^{\beta}((0,1)), \qquad \beta := H_Y - H_{XY},$$

and

$$\left\|\mathfrak{D}^{\mathrm{dist}}_{\beta,1}(\chi\phi^{\sharp}_{v,h})\right\|_E \le C_{t,\alpha}\left(\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}\right\|_E + \|v\|_{\mathfrak{G}_t}\right). \tag{388}$$

Proof. After shrinking $h_{0,\alpha}$ if needed, Theorem 19.4, Proposition 15.3, and Proposition 12.9 all apply with $R = 1$ for every $0 < h \le h_{0,\alpha}$.

Fix such an h and $v \in \mathfrak{G}_t$. Set

$$\Phi^{\chi}_{v,h} := \chi\phi^{\sharp}_{v,h} \in E, \qquad \Psi^{\chi}_{v,h} := \chi\psi^{\sharp}_{v,h} \in E.$$

By Definition 33.14 and the identity

$$\mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} = \mathfrak{A}_{Y,t,h,\alpha}\Psi^{\chi}_{v,h} - \chi\,\mathfrak{A}_{Y,t,h,\alpha}\psi^{\sharp}_{v,h},$$

one has

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{A}_{XY,t,h,\alpha}\Phi^{\chi}_{v,h} - \mathfrak{A}_{Y,t,h,\alpha}\Psi^{\chi}_{v,h} + \mathfrak{W}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} \qquad \text{in } E.$$

Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^{\lambda}_{-,1}$, using Proposition 15.3(i) for $\Psi^{\chi}_{v,h}$ and Proposition 12.9 for the commutator term, yields

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{XY,t,h,\alpha}\Phi^{\chi}_{v,h}) = \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} + \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}\Psi^{\chi}_{v,h} - \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} \tag{389}$$

in $\mathcal{D}'((0,1))$.

The first term on the right-hand side belongs to E by (387). The other two terms also belong to E, because

$$\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}, \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \in \mathcal{L}(E)$$

for small h. Hence the right-hand side of (389) belongs to E.

Now apply the distributional source decomposition (217) from Theorem 19.4 with $R = 1$ to $\Phi^{\chi}_{v,h}$. This gives

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{A}_{XY,t,h,\alpha}\Phi^{\chi}_{v,h}) = \frac{L_Y(t,t)\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}\mathfrak{D}^{\mathrm{dist}}_{\beta,1}\Phi^{\chi}_{v,h} + \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}\Phi^{\chi}_{v,h}$$

in $\mathcal{D}'((0,1))$. Since $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1} \in \mathcal{L}(E)$ and $\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}$ is boundedly invertible for small h, (389) implies that

$$\mathfrak{D}^{\mathrm{dist}}_{\beta,1}\Phi^{\chi}_{v,h} \in E.$$

Lemma 19.2(i) therefore yields

$$\chi\phi^{\sharp}_{v,h} \in H^{\beta}((0,1)).$$

Moreover, the same identity gives

$$\left\|\mathfrak{D}^{\mathrm{dist}}_{\beta,1}\Phi^{\chi}_{v,h}\right\|_E \le C_{t,\alpha}\left(\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}\right\|_E + \|\Psi^{\chi}_{v,h}\|_E + \|\psi^{\sharp}_{v,h}\|_E + \|\Phi^{\chi}_{v,h}\|_E\right).$$

Now $\|\Phi^{\chi}_{v,h}\|_E \le \|\phi^{\sharp}_{v,h}\|_E = \|v\|_{H_t} \le \|v\|_{\mathfrak{G}_t}$, while

$$\|\psi^{\sharp}_{v,h}\|_E \le \varepsilon_{h,\alpha}^{-\beta} h^{\beta} \|\mathcal{S}_t v\|_{H_t} = h^{\alpha\beta}\|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t} \le \frac{\nu_Y}{\eta}\|v\|_{\mathfrak{G}_t},$$

and the same bound holds for $\|\Psi^{\chi}_{v,h}\|_E$. This proves (388). □

Corollary 33.17 (Reduction to a single preconjugated source discrepancy)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^{\infty}([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Assume that there exist

$$C_{\mathrm{pre},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$,

$$\left\| \frac{1}{\Gamma(\lambda)} \mathfrak{D}^{\lambda}_{-,1} \mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} \right\|_E \le C_{\mathrm{pre},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}, \qquad \lambda := H_Y + \frac{1}{2}. \tag{390}$$

Then the conclusion of Corollary 33.9 holds. In particular, $\mathcal{S}_t$ has h^{α}-stable boundary cutoffs at time t, and the screening conclusions (466)– (467) *hold.*

Proof. After shrinking $h_{0,\alpha}$ if needed, Proposition 12.9 applies for every $0 < h \le h_{0,\alpha}$, and Proposition 33.16 applies as well. Fix such an h and $v \in \mathfrak{G}_t$. By Proposition 33.16 and (390),

$$\chi\phi^{\sharp}_{v,h} \in H^{\beta}((0,1)), \qquad \beta := H_Y - H_{XY}.$$

By Proposition 33.15,

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^{\sharp}_{v,h}, \chi\psi^{\sharp}_{v,h}) = \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} - \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}$$

in E.

By (143), there exists $C_t < \infty$, depending only on t and the fixed cutoff χ, such that

$$\left\| \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi} \right\|_{\mathcal{L}(E)} \le C_t.$$

Also, as in the proof of Corollary 33.13,

$$\|\psi^{\sharp}_{v,h}\|_E \le \varepsilon_{h,\alpha}^{-\beta} h^{\beta}\|\mathcal{S}_t v\|_{H_t} = h^{\alpha\beta}\|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}.$$

Therefore

$$\left\| \mathfrak{Q}^{\mathrm{loc}}_{Y,t,h,\alpha,\chi}\psi^{\sharp}_{v,h} \right\|_E \le C_t \frac{\nu_Y}{\eta} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}.$$

Combining this with (390) gives

$$\left\| \mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,1}(\chi\phi^{\sharp}_{v,h}, \chi\psi^{\sharp}_{v,h}) \right\|_E \le C'_{t,\alpha} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t},$$

for some $C'_{t,\alpha} < \infty$. Corollary 33.9 now gives the conclusion. □

Definition 33.18 (Separated old-tail pieces on the unit interval)**.** Under the hypotheses of Proposition 33.5, set

$$\varepsilon := \varepsilon_{h,\alpha} = h^{1-\alpha}, \qquad \lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}.$$

For $v \in \mathfrak{G}_t$ and $0 < h \le t$, define

$$\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} := \varepsilon^{H_{XY}} \mathfrak{l}_{\varepsilon}\left[\mathbf{1}_{[0,\varepsilon^{-1}]} \int_{\varepsilon^{-1}}^{t/h} (q-r)^{\lambda-1} L_Y(t-hr, t-hq)\Psi_{v,h}(q)\,dq \right], \tag{391}$$

$$\vartheta^{\mathrm{old}}_{XY,v,h,\alpha} := \varepsilon^{H_{XY}} \mathfrak{l}_{\varepsilon}\left[\mathbf{1}_{[0,\varepsilon^{-1}]} \int_{\varepsilon^{-1}}^{t/h} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq)\Phi_{v,h}(q)\,dq \right]. \tag{392}$$

Proposition 33.19 (Full-h source realization of the preconjugated discrepancy)**.** *Assume the hypotheses of Proposition 33.5. Let $v \in \mathfrak{G}_t$ and let $0 < h \le t$ satisfy $h^\alpha \le t$. With $\widetilde{\chi}_{h,\alpha}$ from Definition 33.4, define the full-h source discrepancy by*

$$\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} := \mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}) - \widetilde{\chi}_{h,\alpha}\,\mathfrak{V}^{\mathrm{full}}_{XY,t,h}\Phi_{v,h} \in L^2(\mathbb{R}_+). \tag{393}$$

Then:

(i) $\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha}$ is supported in $[0,\varepsilon_{h,\alpha}^{-1}]$;

(ii) one has

$$\omega^{\mathrm{old}}_{v,h,\alpha} = \vartheta^{\mathrm{old}}_{Y,v,h,\alpha} - \vartheta^{\mathrm{old}}_{XY,v,h,\alpha} \qquad \textit{in } E; \tag{394}$$

(iii) one has the exact scaled source identity

$$\varepsilon_{h,\alpha}^{H_{XY}}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^{\sharp}_{v,h} - \chi\,\vartheta^{\mathrm{old}}_{XY,v,h,\alpha} \qquad \textit{in } E; \tag{395}$$

(iv) consequently,

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \varepsilon_{h,\alpha}^{H_{XY}}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} + \chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} \qquad \textit{in } E. \tag{396}$$

Proof. Set $\varepsilon := \varepsilon_{h,\alpha}$. For $r > \varepsilon^{-1}$, one has $\widetilde{\chi}_{h,\alpha}(r) = 0$, and for $q \ge r$ also $\widetilde{\chi}_{h,\alpha}(q) = 0$. Hence both terms in (393) vanish, proving (i).

Item (ii) follows from (360), (363), and Definitions (391)–(392).

For $0 \le r \le \varepsilon^{-1}$, expanding (393) gives

$$\begin{aligned}\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha}(r) = &\int_r^{\varepsilon^{-1}} (q-r)^{\mu-1}L_{XY}(t-hr,t-hq)\big(\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)\big)\Phi_{v,h}(q)\,dq \\ &- \widetilde{\chi}_{h,\alpha}(r)\int_{\varepsilon^{-1}}^{t/h}(q-r)^{\mu-1}L_{XY}(t-hr,t-hq)\Phi_{v,h}(q)\,dq.\end{aligned}$$

Applying $\varepsilon^{H_{XY}}\mathfrak{l}_\varepsilon$, using Proposition 12.12 and the definition (392), yields (395).

Finally, combining (395) with (394) gives

$$\begin{aligned}\varepsilon^{H_{XY}}\mathfrak{l}_\varepsilon\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} + \chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} &= \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^{\sharp}_{v,h} - \chi\,\vartheta^{\mathrm{old}}_{XY,v,h,\alpha} + \chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} \\ &= \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^{\sharp}_{v,h} + \chi\,\omega^{\mathrm{old}}_{v,h,\alpha},\end{aligned}$$

which is exactly $\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}$ by (384). This proves (396). □

Corollary 33.20 (Reduction to full-h source discrepancy plus target old tail)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. Fix*

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \ \textit{on } [0,1-\delta_*]$$

for some $\delta_* \in (0,1)$, and let $\widetilde{\chi}_{h,\alpha}$ be the associated moving micro cutoff. Assume that there exist

$$C_{\mathrm{src},\alpha} < \infty, \qquad C_{\mathrm{tgt},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$,

$$\left\| \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\Big(\varepsilon_{h,\alpha}^{H_{XY}}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha}\Big)\right\|_E \le C_{\mathrm{src},\alpha}h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}, \tag{397}$$

$$\left\| \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\big(\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha}\big)\right\|_E \le C_{\mathrm{tgt},\alpha}h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}, \qquad \lambda := H_Y + \frac{1}{2}. \tag{398}$$

Then the conclusion of Corollary 33.17 holds. In particular, $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and the screening conclusions (466)– (467) *hold.*

Proof. By Proposition 33.19,

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \varepsilon_{h,\alpha}^{H_{XY}}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} + \chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} \qquad \text{in } E.$$

Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^{\lambda}_{-,1}$ and using (397) together with (398) yields

$$\left\| \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}\right\|_E \le (C_{\mathrm{src},\alpha} + C_{\mathrm{tgt},\alpha})h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}.$$

Corollary 33.17 now gives the conclusion. □

Definition 33.21 (Flat outer-edge profile for the endpoint cutoff)**.** Let

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$. We say that χ is *flat at the outer edge* if

$$\chi^{(m)}(1) = 0 \qquad \text{for every } m \in \mathbb{N} \cup \{0\}.$$

Definition 33.22 (Full-h target old-tail commutator)**.** Assume the hypotheses of Proposition 33.5, and let χ be as in Definition 33.21. For $v \in \mathfrak{G}_t$ and $0 < h \le t$, define

$$(\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha})(r) := \int_{\varepsilon_{h,\alpha}^{-1}}^{t/h} (q-r)^{\lambda-1} (\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)) L_Y(t-hr, t-hq) \Psi_{v,h}(q)\, dq, \tag{399}$$

where

$$\lambda := H_Y + \frac{1}{2}.$$

Proposition 33.23 (Exact full-h commutator representation of the target old tail)**.** *Assume the hypotheses of Definition 33.22. Then:*

(i) $\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}$ *is supported in* $[0, \varepsilon_{h,\alpha}^{-1}]$*;*

(ii) one has the exact identity

$$\chi\, \vartheta^{\mathrm{old}}_{Y,v,h,\alpha} = -\varepsilon_{h,\alpha}^{H_{XY}} \mathfrak{l}_{\varepsilon_{h,\alpha}} \mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha} \qquad \text{in } E. \tag{400}$$

Proof. Let $\varepsilon := \varepsilon_{h,\alpha}$. Because $\widetilde{\chi}_{h,\alpha}$ is supported in $[0, \varepsilon^{-1}]$, one has $\widetilde{\chi}_{h,\alpha}(r) = 0$ for $r > \varepsilon^{-1}$, and the integrand in (399) also vanishes for $r > \varepsilon^{-1}$. This proves (i).

Now fix $0 \le r \le \varepsilon^{-1}$. Since $\widetilde{\chi}_{h,\alpha}(q) = 0$ for $q \ge \varepsilon^{-1}$, the definition (399) reduces to

$$\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}(r) = -\widetilde{\chi}_{h,\alpha}(r) \int_{\varepsilon^{-1}}^{t/h} (q-r)^{\lambda-1} L_Y(t-hr, t-hq) \Psi_{v,h}(q)\, dq.$$

Applying $\varepsilon^{H_{XY}} \mathfrak{l}_\varepsilon$ and using

$$(\mathfrak{l}_\varepsilon \widetilde{\chi}_{h,\alpha})(u) = \chi(u) \qquad \text{for } u \in [0,1]$$

gives exactly (400), by (391). □

Lemma 33.24 (Crossing-region quotient bounds for the flat micro cutoff)**.** *Assume the hypotheses of Definition 33.22, and suppose in addition that χ is flat at the outer edge in the sense of Definition 33.21. Set*

$$\Sigma_{h,\alpha} := \left\{ 0 \le r \le \varepsilon_{h,\alpha}^{-1} \le q \le \frac{t}{h} \right\}.$$

Define on $\Sigma_{h,\alpha}$

$$\delta^{\mathrm{old}}_{Y,h,\alpha}(r,q) := \frac{\widetilde{\chi}_{h,\alpha}(q) - \widetilde{\chi}_{h,\alpha}(r)}{q-r}.$$

Then there exists $C_\chi < \infty$, depending only on χ, such that

$$\left\| \delta^{\mathrm{old}}_{Y,h,\alpha} \right\|_{C^1(\Sigma_{h,\alpha})} \le C_\chi \varepsilon_{h,\alpha} \qquad \text{for all } 0 < h \le t. \tag{401}$$

Proof. Let $\varepsilon := \varepsilon_{h,\alpha}$. Since $q \ge \varepsilon^{-1}$, one has $\widetilde{\chi}_{h,\alpha}(q) = 0$, so

$$\delta^{\mathrm{old}}_{Y,h,\alpha}(r,q) = -\frac{\widetilde{\chi}_{h,\alpha}(r)}{q-r} = -\frac{\chi(\varepsilon r)}{q-r} \qquad \text{on } \Sigma_{h,\alpha}.$$

Because χ is flat at 1, Taylor's theorem at the outer edge gives

$$|\chi(x)| \le C_\chi (1-x)^2, \qquad |\chi'(x)| \le C_\chi (1-x) \qquad \text{for } x \in [0,1].$$

Also, for $(r,q) \in \Sigma_{h,\alpha}$,

$$q - r \ge \varepsilon^{-1} - r = \frac{1 - \varepsilon r}{\varepsilon}.$$

Therefore

$$\left| \delta^{\mathrm{old}}_{Y,h,\alpha}(r,q) \right| \le C_\chi \frac{(1-\varepsilon r)^2}{(1-\varepsilon r)/\varepsilon} \le C_\chi \varepsilon.$$

Next,

$$\partial_q \delta^{\mathrm{old}}_{Y,h,\alpha}(r,q) = \frac{\chi(\varepsilon r)}{(q-r)^2},$$

so the same bounds imply

$$\left|\partial_q \delta^{\mathrm{old}}_{Y,h,\alpha}(r,q)\right| \le C_\chi \frac{(1-\varepsilon r)^2}{((1-\varepsilon r)/\varepsilon)^2} \le C_\chi \varepsilon^2 \le C_\chi \varepsilon.$$

Finally,

$$\partial_r \delta^{\mathrm{old}}_{Y,h,\alpha}(r,q) = \frac{-\varepsilon\chi'(\varepsilon r)(q-r) - \chi(\varepsilon r)}{(q-r)^2},$$

hence

$$\begin{aligned}\left|\partial_r \delta^{\mathrm{old}}_{Y,h,\alpha}(r,q)\right| &\le \frac{\varepsilon|\chi'(\varepsilon r)|(q-r) + |\chi(\varepsilon r)|}{(q-r)^2} \\ &\le C_\chi \frac{\varepsilon(1-\varepsilon r)\cdot(1-\varepsilon r)/\varepsilon + (1-\varepsilon r)^2}{((1-\varepsilon r)/\varepsilon)^2} \\ &\le C_\chi \varepsilon^2 \le C_\chi \varepsilon.\end{aligned}$$

This proves (401). □

Proposition 33.25 (α-scale realization of the target old tail)**.** *Assume the hypotheses of Definition 33.22, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Set*

$$\beta := H_Y - H_{XY}, \qquad \varepsilon := \varepsilon_{h,\alpha} = h^{1-\alpha}, \qquad R_{h,\alpha} := th^{-\alpha}.$$

Define the extended α-scale tail trace by

$$\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}(q) := \varepsilon^{-(\beta+\frac{1}{2})}\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\mathbf{1}_{[1,R_{h,\alpha}]}(q), \qquad q \ge 0. \tag{402}$$

Then:

(i) one has

$$\left\|\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}\right\|_{L^2(\mathbb{R}_+)} \le \varepsilon^{-\beta}\left\|\Psi_{v,h}\right\|_{L^2([\varepsilon^{-1},t/h])} \le \frac{\nu_Y}{\eta} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}; \tag{403}$$

(ii) if

$$(\mathfrak{W}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\Theta)(u) := \chi(u)\int_1^{R_{h,\alpha}} (q-u)^{\lambda-1} L_Y(t-h^\alpha u, t-h^\alpha q)\Theta(q)\,dq, \qquad 0 \le u \le 1, \tag{404}$$

then

$$\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} = \varepsilon^{-1/2}\mathfrak{W}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha} \qquad \text{in } E; \tag{405}$$

(iii) for every sufficiently small $h > 0$, there exists

$$\mathfrak{Q}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi} \in \mathcal{L}(L^2([1,R_{h,\alpha}]), E)$$

such that for every $\Theta \in L^2([1,R_{h,\alpha}])$,

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_{-,1}(\mathfrak{W}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\Theta) = \mathfrak{Q}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\Theta \qquad \text{in } E, \tag{406}$$

and there exists $C_{t,\chi} < \infty$, independent of h, such that

$$\left\|\mathfrak{Q}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\right\|_{\mathcal{L}(L^2([1,R_{h,\alpha}]),E)} \le C_{t,\chi}. \tag{407}$$

Consequently,

$$\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_{-,1}(\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha})\right\|_E \le C_{t,\chi}\frac{\nu_Y}{\eta}h^{\alpha\beta-\frac{1-\alpha}{2}}\|v\|_{\mathfrak{G}_t}. \tag{408}$$

Proof. For item (i), the change of variables $q = s/\varepsilon$ gives

$$\left\|\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}\right\|^2_{L^2(\mathbb{R}_+)} = \varepsilon^{-2\beta-1}\int_1^{R_{h,\alpha}}\left|\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\right|^2 dq$$
$$= \varepsilon^{-2\beta}\int_{\varepsilon^{-1}}^{t/h}|\Psi_{v,h}(s)|^2\, ds \le \varepsilon^{-2\beta}h^{2\beta}\|\mathcal{S}_t v\|^2_{H_t}.$$

Since $\varepsilon = h^{1-\alpha}$, this gives (403).

To prove item (ii), let $u \in [0,1]$. Using (391), (349), and the change of variables $q = s/\varepsilon$, we obtain

$$\chi(u)\vartheta^{\mathrm{old}}_{Y,v,h,\alpha}(u) = \chi(u)\varepsilon^{H_{XY}-\frac12}\int_{\varepsilon^{-1}}^{t/h}\left(q-\frac{u}{\varepsilon}\right)^{\lambda-1}L_Y\left(t-h\frac{u}{\varepsilon}, t-hq\right)\Psi_{v,h}(q)\,dq$$
$$= \chi(u)\varepsilon^{H_{XY}-\frac12}\int_1^{R_{h,\alpha}}\left(\frac{q-u}{\varepsilon}\right)^{\lambda-1}L_Y(t-h^\alpha u, t-h^\alpha q)\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\frac{dq}{\varepsilon}$$
$$= \varepsilon^{H_{XY}-\lambda-\frac12}\chi(u)\int_1^{R_{h,\alpha}}(q-u)^{\lambda-1}L_Y(t-h^\alpha u, t-h^\alpha q)\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\, dq.$$

Since $\lambda = H_Y + \frac12$ and $\beta = H_Y - H_{XY}$,

$$\varepsilon^{H_{XY}-\lambda-\frac12}\Psi_{v,h}\left(\frac{q}{\varepsilon}\right) = \varepsilon^{-1/2}\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}(q).$$

This proves (405).

For item (iii), extend χ by zero to a function

$$\overline{\chi} \in C^\infty([0,\infty)),$$

which is possible because χ is flat at the outer edge. For $q \in [1, R_{h,\alpha}]$, define

$$a_{Y,t,h,\alpha,\chi}(u,q) := \overline{\chi}(u)L_Y(t-h^\alpha u, t-h^\alpha q), \qquad 0 \le u \le q.$$

Fix $\Theta \in L^2([1,R_{h,\alpha}])$. Exactly as in the proof of Proposition 17.2, Fubini's theorem gives

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}(\mathfrak{W}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\Theta)(u) = \int_1^{R_{h,\alpha}}K^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}(u,q)\Theta(q)\,dq$$

for almost every $u \in [0,1]$, where

$$K^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}(u,q) := -\frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}\int_0^1 s^{-\lambda}(1-s)^\lambda\partial_1 a_{Y,t,h,\alpha,\chi}(u+(q-u)s, q)\,ds.$$

This defines the operator $\mathfrak{Q}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}$.

To bound its norm, split

$$\partial_1 a_{Y,t,h,\alpha,\chi} = \overline{\chi}'(\cdot)L_Y(t-h^\alpha\cdot, t-h^\alpha q) - h^\alpha\overline{\chi}(\cdot)\partial_1 L_Y(t-h^\alpha\cdot, t-h^\alpha q).$$

For the first term, the support of $\overline{\chi}'$ is contained in $[0,1]$, so the integral in s is effectively restricted to

$$0 \le s \le \rho(u,q) := \frac{1-u}{q-u}.$$

Using the change of variables $x = u + (q-u)s$, one obtains

$$\left|K^{(1)}_{Y,t,h,\alpha,\chi}(u,q)\right| \le C_{t,\chi}(q-u)^{-1}\int_0^{\rho(u,q)} s^{-\lambda}\,ds \le C_{t,\chi}(1-u)^{1-\lambda}(q-u)^{-(2-\lambda)}.$$

Since $2-\lambda > 1$, this kernel is Hilbert–Schmidt on $[0,1]\times[1,\infty)$, hence the corresponding operator is bounded from $L^2([1,R_{h,\alpha}])$ to E, uniformly in h.

For the second term,

$$\left|K^{(2)}_{Y,t,h,\alpha,\chi}(u,q)\right| \le C_{t,\chi}h^\alpha\int_0^{\rho(u,q)} s^{-\lambda}\,ds \le C_{t,\chi}h^\alpha(1-u)^{1-\lambda}(q-u)^{-(1-\lambda)}.$$

Its Hilbert–Schmidt norm is bounded by

$$\begin{aligned}\left\|K^{(2)}_{Y,t,h,\alpha,\chi}\right\|^2_{L^2([0,1]\times[1,R_{h,\alpha}])} &\le C_{t,\chi}h^{2\alpha}\int_0^1(1-u)^{2(1-\lambda)}\int_1^{R_{h,\alpha}}(q-u)^{-2(1-\lambda)}\,dq\,du\\ &\le C_{t,\chi}h^{2\alpha}R_{h,\alpha}^{2\lambda-1}\le C_{t,\chi},\end{aligned}$$

because $2\lambda-1=2H_Y\in(0,1)$ and $R_{h,\alpha}=th^{-\alpha}$. Thus the second operator is also bounded uniformly in h. This proves (407).

Finally, combining (405), (406), (407), and (403) gives

$$\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\big(\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha}\big)\right\|_E\le\varepsilon^{-1/2}\left\|\mathfrak{O}^{\mathrm{old},\alpha}_{Y,t,h,\alpha,\chi}\right\|\left\|\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}\right\|_{L^2(\mathbb{R}_+)}\le C_{t,\chi}\frac{\nu_Y}{\eta}h^{\alpha\beta-\frac{1-\alpha}{2}}\|v\|_{\mathfrak{G}_t},$$

which is (408). □

Corollary 33.26 (Explicit bound for the flat-edge target commutator)**.** *Under the hypotheses of Proposition 33.25,*

$$\left\|\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\Big(\varepsilon^{H_{XY}}_{h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}\Big)\right\|_E\le C_{t,\chi}\frac{\nu_Y}{\eta}h^{\alpha\beta-\frac{1-\alpha}{2}}\|v\|_{\mathfrak{G}_t},\tag{409}$$

where $\lambda=H_Y+\frac12$.

Proof. By Proposition 33.23,

$$\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha}=-\varepsilon^{H_{XY}}_{h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}\qquad\text{in } E.$$

Taking $(\Gamma(\lambda))^{-1}\mathfrak{D}^{\lambda}_{-,1}$, then applying (408), gives (409). □

Remark 33.27 (Limitation of the isolated target estimate)*.* Proposition 33.25 gives an explicit α-scale realization of the target old tail, but it also exposes the obstruction. After the first full-h dilation and the second α-scale zoom, the target remainder carries an unavoidable prefactor $\varepsilon^{-1/2}_{h,\alpha}=h^{-(1-\alpha)/2}$. So the separate target-side route produces, at best, the weaker scale $h^{\alpha\beta-(1-\alpha)/2}$. This is why the argument cannot close the endpoint by bounding the target old tail in isolation: the governing endpoint has to preserve the source-target coupling encoded by $\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}$ or, equivalently, by the corrected full-h realization $\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}$.

Definition 33.28 (Full-h truncated-target realization of the preconjugated discrepancy)**.** Assume the hypotheses of Proposition 33.5. For $v\in\mathfrak{G}_t$, set

$$\lambda:=H_Y+\frac12,\qquad\mu:=H_{XY}+\frac12,\qquad\varepsilon:=\varepsilon_{h,\alpha}=h^{1-\alpha},$$

and for $0<h\le t$, define

$$\begin{aligned}\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}=\mathbf{1}_{[0,\varepsilon^{-1}]}\Bigg[&\mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h})\\ &-\widetilde{\chi}_{h,\alpha}(\cdot)\int_{\cdot}^{\varepsilon^{-1}}\Big((q-\cdot)^{\lambda-1}L_Y(t-h\cdot,t-hq)\Psi_{v,h}(q)\Big)\,dq\Bigg]\in L^2(\mathbb{R}_+).\end{aligned}\tag{410}$$

Proposition 33.29 (Exact full-h realization of the preconjugated discrepancy)**.** *Assume the hypotheses of Definition 33.28. Then:*

(i) $\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}$ *is supported in* $[0,\varepsilon^{-1}_{h,\alpha}]$*;*

(ii) one has

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha}=\varepsilon^{H_{XY}}_{h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}\qquad\textit{in } E.\tag{411}$$

Proof. Let $\varepsilon:=\varepsilon_{h,\alpha}$. Since $\widetilde{\chi}_{h,\alpha}$ is supported in $[0,\varepsilon^{-1}]$, the second term in (410) vanishes for $r>\varepsilon^{-1}$. The first term also vanishes there because $\widetilde{\chi}_{h,\alpha}\Phi_{v,h}$ is supported in $[0,\varepsilon^{-1}]$ and the full Volterra integral starts at $q=r$. This proves (i).

By Proposition 14.5,

$$\mathfrak{A}_{XY,t,h,\alpha}(\chi\phi^{\sharp}_{v,h})=\varepsilon^{H_{XY}}\mathfrak{l}_{\varepsilon}\mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h}),$$

because $\widetilde{\chi}_{h,\alpha}\Phi_{v,h}$ is supported in $[0,\varepsilon^{-1}]$.

For the target term, using $\psi^\sharp_{v,h} = \varepsilon^{-\beta}\mathfrak{l}_\varepsilon \Psi_{v,h}$, $\lambda = H_Y + \frac{1}{2}$, $\beta = H_Y - H_{XY}$, and the change of variables $q = \varepsilon^{-1}v$, one gets for $u \in [0,1]$,

$$\begin{aligned}
\chi(u)\mathfrak{A}_{Y,t,h,\alpha}\psi^\sharp_{v,h}(u) &= \chi(u)\int_u^1 (v-u)^{\lambda-1} L_Y(t-h^\alpha u, t-h^\alpha v)\psi^\sharp_{v,h}(v)\,dv \\
&= \varepsilon^{H_Y-\beta}\chi(u)\int_{\varepsilon^{-1}u}^{\varepsilon^{-1}} (q-\varepsilon^{-1}u)^{\lambda-1} L_Y(t-h\varepsilon^{-1}u, t-hq)\Psi_{v,h}(q)\,dq \\
&= \varepsilon^{H_{XY}}\mathfrak{l}_\varepsilon\left[\widetilde{\chi}_{h,\alpha}(r)\int_r^{\varepsilon^{-1}} (q-r)^{\lambda-1} L_Y(t-hr, t-hq)\Psi_{v,h}(q)\,dq\right](u).
\end{aligned}$$

Subtracting the two identities proves (411). $\square$

Proposition 33.30 (Exact split of the corrected full-h realization)**.** *Assume the hypotheses of Definition 33.28. Then*

$$\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha} = \mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha} - \mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha} \qquad \text{in } L^2(\mathbb{R}_+), \tag{412}$$

where $\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha}$ is the full-h source discrepancy from (393) *and $\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}$ is the target old-tail commutator from* (399)*.*

Proof. Let $\varepsilon := \varepsilon_{h,\alpha}$. For $r > \varepsilon^{-1}$, all three terms in (412) vanish, so it is enough to work on $0 \le r \le \varepsilon^{-1}$. By definition,

$$\begin{aligned}
\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}(r) &= \mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h})(r) \\
&\quad - \widetilde{\chi}_{h,\alpha}(r)\int_r^{\varepsilon^{-1}} (q-r)^{\lambda-1} L_Y(t-hr, t-hq)\Psi_{v,h}(q)\,dq.
\end{aligned}$$

Adding and subtracting

$$\widetilde{\chi}_{h,\alpha}(r)\int_r^{t/h} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq)\Phi_{v,h}(q)\,dq$$

and using the exact full-h identity (350) gives

$$\begin{aligned}
\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}(r) &= \mathfrak{V}^{\mathrm{full}}_{XY,t,h}(\widetilde{\chi}_{h,\alpha}\Phi_{v,h})(r) \\
&\quad - \widetilde{\chi}_{h,\alpha}(r)\int_r^{t/h} (q-r)^{\mu-1} L_{XY}(t-hr, t-hq)\Phi_{v,h}(q)\,dq \\
&\quad + \widetilde{\chi}_{h,\alpha}(r)\int_{\varepsilon^{-1}}^{t/h} (q-r)^{\lambda-1} L_Y(t-hr, t-hq)\Psi_{v,h}(q)\,dq.
\end{aligned}$$

The first two lines are exactly $\mathfrak{Z}^{\mathrm{src}}_{v,h,\alpha}(r)$, while the last line is $-\mathfrak{C}^{\mathrm{old}}_{Y,v,h,\alpha}(r)$ by (399). This proves (412). $\square$

Corollary 33.31 (Endpoint reduction to the corrected full-h realization)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1. If there exist*

$$C_{\mathrm{pair},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$,

$$\left\| \frac{1}{\Gamma(\lambda)}\mathfrak{D}^\lambda_{-,1}\left(\varepsilon^{H_{XY}}_{h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha}\right)\right\|_E \le C_{\mathrm{pair},\alpha} h^{\alpha\beta}\|v\|_{\mathfrak{G}_t}, \qquad \lambda := H_Y + \frac{1}{2}, \tag{413}$$

then the conclusion of Corollary 33.17 holds. In particular, $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and the screening conclusions (466)– (467) *hold.*

Proof. By Proposition 33.29,

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \varepsilon^{H_{XY}}_{h,\alpha}\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha} \qquad \text{in } E.$$

Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^\lambda_{-,1}$ and using (413) gives (390). Corollary 33.17 now gives the conclusion. $\square$

Definition 33.32 (Coupled α-scale endpoint traces)**.** Assume the hypotheses of Definition 33.28. Set

$$\beta := H_Y - H_{XY}, \qquad \varepsilon := \varepsilon_{h,\alpha} = h^{1-\alpha}, \qquad R_{h,\alpha} := th^{-\alpha}.$$

Define

$$\widehat{E}^X_{h,\alpha} := L^2([0, R_{h,\alpha}]), \qquad \widehat{E}^Y_{h,\alpha} := L^2([0, R_{h,\alpha}]), \qquad \widehat{E}^{Y,\mathrm{old}}_{h,\alpha} := L^2([1, R_{h,\alpha}]).$$

For $v \in \mathfrak{G}_t$, define the extended α-scale source trace and the α-scale target traces by

$$\widehat{\Phi}_{v,h,\alpha}(q) := \varepsilon^{-1/2}\Phi_{v,h}\left(\frac{q}{\varepsilon}\right)\mathbf{1}_{[0,R_{h,\alpha}]}(q), \tag{414}$$

$$\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}(q) := \varepsilon^{-(\beta+\frac{1}{2})}\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\mathbf{1}_{[0,R_{h,\alpha}]}(q), \tag{415}$$

$$\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}(q) := \varepsilon^{-(\beta+\frac{1}{2})}\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)\mathbf{1}_{[1,R_{h,\alpha}]}(q). \tag{416}$$

Finally, define the admissible coupled α-scale trace class by

$$\widehat{\mathcal{A}}^{\mathrm{pair}}_{t,h,\alpha} := \left\{(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}) : v \in \mathfrak{G}_t\right\} \subset \widehat{E}^X_{h,\alpha} \times \widehat{E}^{Y,\mathrm{old}}_{h,\alpha}. \tag{417}$$

Proposition 33.33 (Exact full α-scale pair on the growing window)**.** *Assume the hypotheses of Definition 33.32. Then for every*

$$v \in \mathfrak{G}_t, \qquad 0 < h \le t, \qquad h^\alpha \le t,$$

the pair

$$(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha})$$

belongs to the exact fixed-window graph

$$\Gamma^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}.$$

Equivalently,

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}\widehat{\Phi}_{v,h,\alpha} = \widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}. \tag{418}$$

Moreover,

$$\left\|\widehat{\Phi}_{v,h,\alpha}\right\|^2_{E_{R_{h,\alpha}}} + \frac{\eta^2}{\nu_Y^2}\left\|\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}\right\|^2_{E_{R_{h,\alpha}}} \le \|v\|^2_{\mathfrak{G}_t}. \tag{419}$$

Proof. Let $\varepsilon := \varepsilon_{h,\alpha}$ and $R := R_{h,\alpha} = t\varepsilon/h$. By (350),

$$\mathfrak{V}^{\mathrm{full}}_{Y,t,h}\Psi_{v,h} = \mathfrak{V}^{\mathrm{full}}_{XY,t,h}\Phi_{v,h} \qquad \text{in } L^2(\mathbb{R}_+).$$

For $0 \le r \le R$, Proposition 14.5 with the change of variables $q = s/\varepsilon$ gives

$$\begin{aligned}(\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha})(r) &= \int_r^R (q-r)^{H_Y-\frac{1}{2}}L_Y(t-h^\alpha r, t-h^\alpha q)\varepsilon^{-(\beta+\frac{1}{2})}\Psi_{v,h}\left(\frac{q}{\varepsilon}\right)dq \\ &= \varepsilon^{H_{XY}}\left(\mathfrak{V}^{\mathrm{full}}_{Y,t,h}\Psi_{v,h}\right)\left(\frac{r}{\varepsilon}\right),\end{aligned}$$

because $h^\alpha = \varepsilon h$ and $\beta = H_Y - H_{XY}$. The same computation gives

$$(\mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\widehat{\Phi}_{v,h,\alpha})(r) = \varepsilon^{H_{XY}}\left(\mathfrak{V}^{\mathrm{full}}_{XY,t,h}\Phi_{v,h}\right)\left(\frac{r}{\varepsilon}\right).$$

Hence

$$\mathfrak{V}^{\mathrm{mic}}_{Y,t,h,\alpha,R}\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha} = \mathfrak{V}^{\mathrm{mic}}_{XY,t,h,\alpha,R}\widehat{\Phi}_{v,h,\alpha} \qquad \text{in } E_R,$$

so

$$(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}$$

by Definition 16.5. Proposition 18.2 then gives (418).

Finally, by the change of variables $q = s/\varepsilon$,

$$\left\|\widehat{\Phi}_{v,h,\alpha}\right\|_{E_R} = \|\Phi_{v,h}\|_{L^2([0,t/h])} = \|v\|_{H_t},$$

while
$$\left\|\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}\right\|_{E_R} = \varepsilon^{-\beta}\left\|\Psi_{v,h}\right\|_{L^2([0,t/h])} = h^{\alpha\beta}\|\mathcal{S}_t v\|_{H_t} \le \|\mathcal{S}_t v\|_{H_t}.$$
Combining these with the definition of $\|v\|_{\mathfrak{G}_t}$ proves (419). □

Definition 33.34 (Coupled α-scale endpoint operator)**.** Under the hypotheses of Definition 33.32, set
$$\lambda := H_Y + \frac{1}{2}, \qquad \mu := H_{XY} + \frac{1}{2}.$$
For
$$\Theta_X \in \widehat{E}^{X}_{h,\alpha}, \qquad \Theta_Y \in \widehat{E}^{Y,\mathrm{old}}_{h,\alpha},$$
define
$$\mathfrak{W}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\Theta_X,\Theta_Y) \in E = L^2([0,1])$$
by
$$\begin{aligned}(\mathfrak{W}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\Theta_X,\Theta_Y))(u) &:= \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}(\Theta_X|_{[0,1]})(u)\\ &\quad + \varepsilon_{h,\alpha}^{-1/2}\chi(u)\int_1^{R_{h,\alpha}}\Big[(q-u)^{\lambda-1}L_Y(t-h^\alpha u, t-h^\alpha q)\Theta_Y(q)\\ &\qquad - (q-u)^{\mu-1}L_{XY}(t-h^\alpha u, t-h^\alpha q)\Theta_X(q)\Big]\,dq,\end{aligned} \tag{420}$$
for $0 \le u \le 1$. Also define the conjugated distributional realization by
$$\mathfrak{Q}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\Theta_X,\Theta_Y) := \frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{W}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\Theta_X,\Theta_Y) \in \mathcal{D}'((0,1)). \tag{421}$$

Proposition 33.35 (Exact coupled α-scale realization of the preconjugated discrepancy)**.** *Assume the hypotheses of Definition 33.32. Then for every*
$$v \in \mathfrak{G}_t, \qquad 0 < h \le t, \qquad h^\alpha \le t,$$
one has
$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{W}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}) \qquad \textit{in } E. \tag{422}$$
Consequently, in $\mathcal{D}'((0,1))$,
$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{Q}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}). \tag{423}$$

Proof. Let $\varepsilon := \varepsilon_{h,\alpha}$ and $R := R_{h,\alpha}$. By Proposition 33.19,
$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \mathfrak{W}^{\mathrm{loc}}_{XY,t,h,\alpha,\chi}\phi^{\sharp}_{v,h} - \chi\,\vartheta^{\mathrm{old}}_{XY,v,h,\alpha} + \chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha} \qquad \text{in } E.$$
Since
$$\widehat{\Phi}_{v,h,\alpha}|_{[0,1]} = \phi^{\sharp}_{v,h},$$
it remains to identify the two old-tail pieces. For the target contribution, (391), (416), and the change of variables $q = s/\varepsilon$ give
$$\chi\,\vartheta^{\mathrm{old}}_{Y,v,h,\alpha}(u) = \varepsilon^{-1/2}\chi(u)\int_1^R (q-u)^{\lambda-1}L_Y(t-h^\alpha u, t-h^\alpha q)\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}(q)\,dq \tag{424}$$
for $0 \le u \le 1$, because $\lambda = H_Y + \frac{1}{2}$ and $\beta = H_Y - H_{XY}$.

For the source contribution, (392), (414), and the change of variables $q = s/\varepsilon$ give
$$\begin{aligned}\chi(u)\vartheta^{\mathrm{old}}_{XY,v,h,\alpha}(u) &= \chi(u)\varepsilon^{H_{XY}-\frac{1}{2}}\int_{\varepsilon^{-1}}^{t/h}\left(q-\frac{u}{\varepsilon}\right)^{\mu-1}L_{XY}\left(t-h\frac{u}{\varepsilon}, t-hq\right)\Phi_{v,h}(q)\,dq\\ &= \chi(u)\varepsilon^{H_{XY}-\frac{1}{2}}\int_1^R\left(\frac{q-u}{\varepsilon}\right)^{\mu-1}L_{XY}(t-h^\alpha u, t-h^\alpha q)\Phi_{v,h}\left(\frac{q}{\varepsilon}\right)\frac{dq}{\varepsilon}\\ &= \varepsilon^{-1/2}\chi(u)\int_1^R (q-u)^{\mu-1}L_{XY}(t-h^\alpha u, t-h^\alpha q)\widehat{\Phi}_{v,h,\alpha}(q)\,dq,\end{aligned}$$
because $\mu = H_{XY} + \frac{1}{2}$. Combining this with (424) proves (422). Applying $(\Gamma(\lambda))^{-1}\mathfrak{D}^{\lambda}_{-,1}$ gives (423). □

Proposition 33.36 (Exact defect realization of the coupled α-scale endpoint)**.** *Assume the hypotheses of Proposition 33.33, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Let*

$$\overline{\chi}(q) := \begin{cases} \chi(q), & 0 \le q \le 1, \\ 0, & 1 < q \le R_{h,\alpha}. \end{cases}$$

Then $\overline{\chi} \in C^\infty([0, R_{h,\alpha}])$, and for every $v \in \mathfrak{G}_t$ one has

$$\overline{\chi}\,\widehat{\Phi}_{v,h,\alpha} \in H^\beta((0, R_{h,\alpha})), \qquad \overline{\chi}\,\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha} \in E_{R_{h,\alpha}},$$

and

$$\mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}(\overline{\chi}\,\widehat{\Phi}_{v,h,\alpha}, \overline{\chi}\,\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}) = \mathfrak{Q}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}) \qquad \text{in } E. \tag{425}$$

Proof. Set

$$R := R_{h,\alpha}, \qquad \Phi := \widehat{\Phi}_{v,h,\alpha}, \qquad \Psi := \widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}.$$

By Proposition 33.33,

$$(\Phi, \Psi) \in \Gamma^{\mathrm{ex}}_{t,h,\alpha,R}.$$

Since $\overline{\chi} \in C^\infty([0, R])$ and $\Phi \in H^\beta((0, R))$ by Theorem 19.4, one has

$$\overline{\chi}\,\Phi \in H^\beta((0, R)).$$

Because (Φ, Ψ) is an exact fixed-window pair, the proof of Theorem 20.3 applied on the window $[0, R]$ gives

$$\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\overline{\chi}\,\Phi, \overline{\chi}\,\Psi) = \mathfrak{Q}^{\mathrm{com}}_{Y,t,h,\alpha,\overline{\chi}}\Psi - \mathfrak{Q}^{\mathrm{com}}_{XY,t,h,\alpha,\overline{\chi}}\Phi \qquad \text{in } E_R.$$

Restrict now to $r \in [0, 1]$. Since $\overline{\chi}(r) = \chi(r)$, while $\overline{\chi}(q) = 0$ for $q \ge 1$, the two commutator integrals split into the local unit-interval commutator on $[r, 1]$ and the coupled tail contribution on $[1, R]$. Using $\Psi|_{[1,R]} = \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}$, the resulting expression is the distributional definition (421) of

$$\mathfrak{Q}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha})$$

on $E = L^2([0, 1])$. This proves (425). □

Corollary 33.37 (Endpoint reduction to a growing-window exact defect)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. If there exist*

$$C^\flat_{\mathrm{pair},\alpha} < \infty, \qquad h_{0,\alpha} \in (0, t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$,

$$\left\| \mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}(\overline{\chi}\,\widehat{\Phi}_{v,h,\alpha}, \overline{\chi}\,\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}) \right\|_E \le C^\flat_{\mathrm{pair},\alpha} h^{\alpha\beta} \|v\|_{\mathfrak{G}_t}, \tag{426}$$

then the hypothesis of Corollary 33.31 is satisfied. Consequently, the conclusion of Corollary 33.31 follows.

Proof. By Proposition 33.36,

$$\mathfrak{Q}^{\mathrm{pair},\alpha}_{t,h,\alpha,\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha}) = \mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}(\overline{\chi}\,\widehat{\Phi}_{v,h,\alpha}, \overline{\chi}\,\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}) \qquad \text{in } E.$$

Proposition 33.35 identifies the left-hand side with

$$\frac{1}{\Gamma(\lambda)}\mathfrak{D}^{\lambda}_{-,1}\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} \qquad \text{in } \mathcal{D}'((0, 1)),$$

and Proposition 33.29 gives

$$\mathfrak{Y}^{\mathrm{src}}_{v,h,\alpha} = \varepsilon^{H_{XY}}_{h,\alpha}\,\mathfrak{l}_{\varepsilon_{h,\alpha}}\mathfrak{C}^{\mathrm{pair}}_{v,h,\alpha} \qquad \text{in } E.$$

Hence the assumed bound (426) is exactly the hypothesis of Corollary 33.31, which then yields the conclusion. □

Definition 33.38 (Normalized growing-window endpoint traces)**.** Assume the hypotheses of Definition 33.32. For $v \in \mathfrak{G}_t$, set

$$\widehat{Z}^{\mathrm{full}}_{v,h,\alpha} := h^{-\alpha\beta}\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha} \in \widehat{E}^{Y}_{h,\alpha}, \tag{427}$$

and

$$\widehat{Z}^{\mathrm{old}}_{v,h,\alpha} := h^{-\alpha\beta}\widehat{\Psi}^{\mathrm{old}}_{v,h,\alpha} \in \widehat{E}^{Y,\mathrm{old}}_{h,\alpha}. \tag{428}$$

Proposition 33.39 (Normalized growing-window exact pair)**.** *Assume the hypotheses of Definition 33.38. Then for every*

$$v \in \mathfrak{G}_t, \qquad 0 < h \le t, \qquad h^{\alpha} \le t,$$

one has

$$\mathfrak{T}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}\widehat{\Phi}_{v,h,\alpha} = h^{\alpha\beta}\widehat{Z}^{\mathrm{full}}_{v,h,\alpha}, \tag{429}$$

and

$$\left\|\widehat{\Phi}_{v,h,\alpha}\right\|_{E_{R_{h,\alpha}}} + \frac{\eta}{\nu_Y}h^{\alpha\beta}\left\|\widehat{Z}^{\mathrm{full}}_{v,h,\alpha}\right\|_{E_{R_{h,\alpha}}} \le \left(1+\frac{\eta}{\nu_Y}\right)\|v\|_{\mathfrak{G}_t}. \tag{430}$$

In particular,

$$\left\|\widehat{Z}^{\mathrm{full}}_{v,h,\alpha}\right\|_{E_{R_{h,\alpha}}} = \|\mathcal{S}_t v\|_{H_t} \le \frac{\nu_Y}{\eta}\|v\|_{\mathfrak{G}_t}. \tag{431}$$

Proof. Equation (429) is (418) together with (427). The first bound follows from (419) by taking square roots and using

$$\sqrt{a^2+b^2} \le a+b \qquad \text{for } a,b \ge 0.$$

Finally, (431) is exactly the computation already used in the proof of Proposition 33.33:

$$\left\|\widehat{Z}^{\mathrm{full}}_{v,h,\alpha}\right\|_{E_{R_{h,\alpha}}} = h^{-\alpha\beta}\left\|\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}\right\|_{E_{R_{h,\alpha}}} = \|\mathcal{S}_t v\|_{H_t}.$$

□

Definition 33.40 (Normalized growing-window exact defect)**.** Assume the hypotheses of Proposition 33.36. For

$$R \ge 1, \qquad \Phi \in H^{\beta}((0,R)), \qquad Z \in E_R,$$

define

$$\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R,\chi}(\Phi,Z) := h^{-\alpha\beta}\mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\overline{\chi}\,\Phi, h^{\alpha\beta}\overline{\chi}\,Z) \in E, \tag{432}$$

where $\overline{\chi}$ denotes the zero extension of χ from $[0,1]$ to $[0,R]$.

Proposition 33.41 (Endpoint bound is a uniform normalized-defect estimate)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then* (426) *is equivalent to the statement that there exist*

$$C^{\mathrm{norm}}_{\mathrm{pair},\alpha} < \infty, \qquad h_{0,\alpha} \in (0,t]$$

such that for every $0 < h \le h_{0,\alpha}$ and every $v \in \mathfrak{G}_t$,

$$\left\|\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha},\widehat{Z}^{\mathrm{full}}_{v,h,\alpha})\right\|_{E} \le C^{\mathrm{norm}}_{\mathrm{pair},\alpha}\|v\|_{\mathfrak{G}_t}. \tag{433}$$

Proof. By Definition 33.40 and (427),

$$\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha},\widehat{Z}^{\mathrm{full}}_{v,h,\alpha}) = h^{-\alpha\beta}\mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R_{h,\alpha}}(\overline{\chi}\,\widehat{\Phi}_{v,h,\alpha},\overline{\chi}\,\widehat{\Psi}^{\mathrm{full}}_{v,h,\alpha}).$$

Therefore (426) and (433) are the same statement with a different normalization of the left-hand side. □

Definition 33.42 (Normalized frozen growing-window defect)**.** Assume

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

fix $t \in (0,T)$, fix $\alpha \in (0,1)$, let $0 < h \le t$, let $R \ge 1$, and let $\chi \in C^\infty([0,1])$. Extend χ by zero to $[0,R]$, still denoted $\overline{\chi}$. For

$$\Phi \in H^\beta((0,R)), \qquad Z \in E_R,$$

define the normalized frozen localized defect by

$$\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi, Z) := \mathbf{1}_{[0,1]} L_Y(t,t)\Big(\overline{\chi}\, Z - h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,R}(\overline{\chi}\,\Phi)\Big) \in E. \tag{434}$$

Proposition 33.43 (Normalized frozen localized defect is a commutator)**.** *Assume the hypotheses of Definition 33.42. Then if*

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = h^{\alpha\beta} Z,$$

then

$$\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi, Z) = \mathbf{1}_{[0,1]} L_Y(t,t)\, h^{-\alpha\beta}[\overline{\chi}, \mathfrak{T}^{\mathrm{hl}}_{t,R}]\Phi \qquad \textit{in } E. \tag{435}$$

Equivalently, if one defines the normalized frozen defect by

$$\mathfrak{F}^{\mathrm{hl}}_{t,h,\alpha,R}(\Phi, \Psi) := L_Y(t,t)(\Psi - h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi), \qquad \Phi \in H^\beta((0,R)),\ \Psi \in E_R,$$

then

$$\mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{hl}}_{t,h,\alpha,R}(\overline{\chi}\,\Phi, \overline{\chi}\, Z) = \mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi, Z). \tag{436}$$

Proof. Since $\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = h^{\alpha\beta}Z$, one has

$$\overline{\chi}\, Z - h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,R}(\overline{\chi}\,\Phi) = h^{-\alpha\beta}\overline{\chi}\,\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi - h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,R}(\overline{\chi}\,\Phi) = h^{-\alpha\beta}[\overline{\chi}, \mathfrak{T}^{\mathrm{hl}}_{t,R}]\Phi.$$

Multiplying by $L_Y(t,t)$ and restricting to $[0,1]$ gives (435). The identity (436) is simply the definition of $\mathfrak{F}^{\mathrm{hl}}_{t,h,\alpha,R}$. ☐

Definition 33.44 (Finite-window right fractional integral)**.** For $R > 0$, $\lambda \in (0,1)$, and $f \in E_R = L^2([0,R])$, define

$$(\mathfrak{I}^\lambda_{-,R} f)(r) := \frac{1}{\Gamma(\lambda)}\int_r^R (q-r)^{\lambda-1} f(q)\, dq, \qquad 0 \le r \le R. \tag{437}$$

Lemma 33.45 (Finite-window right fractional integral inverts the frozen transfer)**.** *Assume*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}, \qquad \beta := H_Y - H_{XY},$$

fix $t \in (0,T)$, and let $R > 0$. Then:

(i) for every $f \in E_R$,

$$\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\mathfrak{I}^\beta_{-,R} f) = f \qquad \textit{in } E_R; \tag{438}$$

(ii) for every $f \in E_R$, one has

$$\mathfrak{I}^\beta_{-,R} f \in H^\beta((0,R));$$

(iii) for every $f \in E_R$,

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}(\mathfrak{I}^\beta_{-,R} f) = \mathbf{c}_t f \qquad \textit{in } E_R, \tag{439}$$

where $\mathbf{c}_t$ is defined in (82);

(iv) consequently, if $\Phi \in H^\beta((0,R))$ and $Z \in E_R$ satisfy

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = h^{\alpha\beta} Z, \tag{440}$$

then

$$\Phi = \mathbf{c}_t^{-1} h^{\alpha\beta}\mathfrak{I}^\beta_{-,R} Z \qquad \textit{in } H^\beta((0,R)). \tag{441}$$

Proof. For $\lambda, \mu \in (0,1)$ with $\lambda + \mu \le 1$, the same Fubini and Beta-function computation as in Lemma 11.10 gives the finite-window semigroup identity

$$\mathfrak{I}^{\lambda}_{-,R}\mathfrak{I}^{\mu}_{-,R} = \mathfrak{I}^{\lambda+\mu}_{-,R} \qquad \text{on } E_R. \tag{442}$$

Applying (442) with $\lambda = 1-\beta$ and $\mu = \beta$ yields

$$\mathfrak{I}^{1-\beta}_{-,R}\mathfrak{I}^{\beta}_{-,R}f = \mathfrak{I}^{1}_{-,R}f, \qquad (\mathfrak{I}^{1}_{-,R}f)(r) = \int_r^R f(q)\,dq.$$

Since

$$\mathfrak{D}^{\mathrm{dist}}_{\beta,R}g = -\frac{d}{dr}\mathfrak{I}^{1-\beta}_{-,R}g$$

by definition, differentiating the previous identity gives (438). It also follows that $\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\mathfrak{I}^{\beta}_{-,R}f) \in E_R$, so Lemma 19.2(i) gives $\mathfrak{I}^{\beta}_{-,R}f \in H^{\beta}((0,R))$, proving (ii).

Now combine (438) with Lemma 19.2(ii):

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}(\mathfrak{I}^{\beta}_{-,R}f) = \frac{\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\mathfrak{I}^{\beta}_{-,R}f) = \mathbf{c}_t f.$$

This proves (439). Finally, if $\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = h^{\alpha\beta}Z$, then (439) shows that

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Big(\Phi - \mathbf{c}_t^{-1}h^{\alpha\beta}\mathfrak{I}^{\beta}_{-,R}Z\Big) = 0.$$

By Definition 16.2 and the injectivity of $\mathfrak{J}_{H_{XY},R}$ proved in (194), $\mathfrak{T}^{\mathrm{hl}}_{t,R}$ is injective, so (441) follows. ☐

Theorem 33.46 (Uniform growing-window bound for the normalized frozen defect)**.** *Assume the hypotheses of Definition 33.42, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then there exists $C_{t,\chi} < \infty$ such that for every*

$$0 < h \le t, \qquad R \ge 1, \qquad \Phi \in H^{\beta}((0,R)), \qquad Z \in E_R$$

with

$$\mathfrak{T}^{\mathrm{hl}}_{t,R}\Phi = h^{\alpha\beta}Z,$$

one has

$$\Big\|\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z)\Big\|_E \le C_{t,\chi}\|Z\|_{E_R}. \tag{443}$$

For almost every $r \in [0,1]$,

$$(\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z))(r) = L_Y(t,t)\int_r^R K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)Z(q)\,dq, \tag{444}$$

where

$$K^{\mathrm{fr}}_{\beta,\chi,R}(r,q) := \frac{1}{\Gamma(1-\beta)\Gamma(\beta)}\int_0^1 \theta^{-\beta}(1-\theta)^{\beta}\overline{\chi}'(r+\theta(q-r))\,d\theta, \qquad 0 \le r \le 1,\ r \le q \le R, \tag{445}$$

and

$$\Big|K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)\Big| \le C_{\beta,\chi}, \qquad 0 \le r \le q \le 1, \tag{446}$$

$$\Big|K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)\Big| \le C_{\beta,\chi}\left(\frac{1-r}{q-r}\right)^{1-\beta}, \qquad 0 \le r \le 1,\ 1 \le q \le R. \tag{447}$$

Proof. Because χ is flat at the outer edge and vanishes identically on $(1,R]$, its zero extension $\overline{\chi}$ belongs to $C^{\infty}([0,R])$. By Lemma 33.45(iv),

$$\Phi = \mathbf{c}_t^{-1}h^{\alpha\beta}\mathfrak{I}^{\beta}_{-,R}Z.$$

Hence, using Lemma 19.2(ii),

$$\begin{aligned}\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z) &= \mathbf{1}_{[0,1]}L_Y(t,t)\Big(\overline{\chi}\,Z - \frac{\mathbf{c}_t}{\Gamma(1-\beta)}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\overline{\chi}\,\mathbf{c}_t^{-1}\mathfrak{I}^{\beta}_{-,R}Z)\Big)\\ &= \mathbf{1}_{[0,1]}L_Y(t,t)\Big(\overline{\chi}\,Z - \frac{1}{\Gamma(1-\beta)}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\overline{\chi}\,\mathfrak{I}^{\beta}_{-,R}Z)\Big).\end{aligned} \tag{448}$$

Fix $r \in [0,1]$. Since $\overline{\chi}$ is supported in $[0,1]$, Fubini's theorem yields

$$(\mathfrak{I}^{1-\beta}_{-,R}(\overline{\chi}\,\mathfrak{I}^{\beta}_{-,R}Z))(r) = \frac{1}{\Gamma(1-\beta)\Gamma(\beta)}\int_r^1\int_s^R (s-r)^{-\beta}\overline{\chi}(s)(q-s)^{\beta-1}Z(q)\,dq\,ds$$
$$= \int_r^R A_{\beta,\chi,R}(r,q)Z(q)\,dq,$$

where

$$A_{\beta,\chi,R}(r,q) := \frac{1}{\Gamma(1-\beta)\Gamma(\beta)}\int_0^1 \theta^{-\beta}(1-\theta)^{\beta-1}\overline{\chi}(r+\theta(q-r))\,d\theta.$$

Because $\overline{\chi} \in C^\infty([0,R])$, the map $A_{\beta,\chi,R}$ is C^1 on $\{0 \le r \le 1,\ r \le q \le R\}$, and

$$A_{\beta,\chi,R}(r,r) = \overline{\chi}(r).$$

Differentiating under the integral sign gives

$$\partial_r A_{\beta,\chi,R}(r,q) = \frac{1}{\Gamma(1-\beta)\Gamma(\beta)}\int_0^1 \theta^{-\beta}(1-\theta)^{\beta}\overline{\chi}'(r+\theta(q-r))\,d\theta = K^{\mathrm{fr}}_{\beta,\chi,R}(r,q).$$

Therefore

$$\frac{1}{\Gamma(1-\beta)}\mathfrak{D}^{\mathrm{dist}}_{\beta,R}(\overline{\chi}\,\mathfrak{I}^{\beta}_{-,R}Z)(r) = -\frac{d}{dr}\int_r^R A_{\beta,\chi,R}(r,q)Z(q)\,dq$$
$$= \overline{\chi}(r)Z(r) - \int_r^R K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)Z(q)\,dq.$$

Substituting this identity into (448) proves (444).

Now fix $0 \le r \le q \le 1$. Since $\overline{\chi}'$ is bounded on $[0,1]$, (445) gives (446). Next fix $0 \le r \le 1 \le q \le R$. Because $\overline{\chi}' = 0$ on $[1,R]$, the integrand in (445) vanishes unless

$$r + \theta(q-r) \le 1 \qquad \Longleftrightarrow \qquad \theta \le \frac{1-r}{q-r}.$$

Hence

$$\left|K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)\right| \le \frac{\|\overline{\chi}'\|_{L^\infty([0,1])}}{\Gamma(1-\beta)\Gamma(\beta)}\int_0^{\frac{1-r}{q-r}} \theta^{-\beta}(1-\theta)^{\beta}\,d\theta \le C_{\beta,\chi}\left(\frac{1-r}{q-r}\right)^{1-\beta},$$

which proves (447).

Split the operator in (444) into the local part $q \in [r,1]$ and the tail part $q \in [1,R]$. By (446), the local kernel is bounded on the finite triangle $\{0 \le r \le q \le 1\}$, hence Hilbert–Schmidt from $L^2([0,1])$ to E. For the tail part, (447) gives

$$\int_0^1\int_1^R \left|K^{\mathrm{fr}}_{\beta,\chi,R}(r,q)\right|^2 dq\,dr \le C_{\beta,\chi}\int_0^1\int_1^\infty \left(\frac{1-r}{q-r}\right)^{2(1-\beta)} dq\,dr.$$

Because $\beta < \frac{1}{2}$, the last integral is finite:

$$\int_1^\infty (q-r)^{-2(1-\beta)}\,dq = \frac{(1-r)^{2\beta-1}}{1-2\beta},$$

so

$$\int_0^1\int_1^\infty \left(\frac{1-r}{q-r}\right)^{2(1-\beta)} dq\,dr = \frac{1}{1-2\beta}\int_0^1 (1-r)\,dr < \infty.$$

Thus the tail kernel is also Hilbert–Schmidt, with norm bounded independently of $R \ge 1$. The two Hilbert–Schmidt bounds combine to give

$$\left\|\mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z)\right\|_E \le C_{t,\chi}\|Z\|_{E_R},$$

which is (443). $\square$

Corollary 33.47 (Normalized growing-window exact pairs satisfy the frozen bound)**.** *Assume the hypotheses of Proposition 33.39, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then there exists $C_{t,\chi} < \infty$ such that for every $0 < h \le t$ with $h^\alpha \le t$ and every $v \in \mathfrak{G}_t$,*

$$\left\| \mathfrak{C}^{\mathrm{fr}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{Z}^{\mathrm{full}}_{v,h,\alpha}) \right\|_E \le C_{t,\chi} \|v\|_{\mathfrak{G}_t}. \tag{449}$$

Proof. Apply Theorem 33.46 with

$$R = R_{h,\alpha}, \qquad \Phi = \widehat{\Phi}_{v,h,\alpha}, \qquad Z = \widehat{Z}^{\mathrm{full}}_{v,h,\alpha}.$$

The exact relation

$$\mathfrak{T}^{\mathrm{hl}}_{t,R_{h,\alpha}} \widehat{\Phi}_{v,h,\alpha} = h^{\alpha\beta} \widehat{Z}^{\mathrm{full}}_{v,h,\alpha}$$

is Proposition 33.39. The bound (431) then gives

$$\left\| \widehat{Z}^{\mathrm{full}}_{v,h,\alpha} \right\|_{E_{R_{h,\alpha}}} \le \frac{\nu_Y}{\eta} \|v\|_{\mathfrak{G}_t},$$

which proves (449). □

Lemma 33.48 (Uniform head-window bound for the finite-window fractional inverse)**.** *Assume*

$$0 < \beta < \tfrac{1}{2}.$$

Then there exists $C_\beta < \infty$ such that for every $R \ge 1$ and every $f \in E_R$,

$$\left\| \mathbf{1}_{[0,1]} \mathfrak{I}^{\beta}_{-,R} f \right\|_E \le C_\beta \|f\|_{E_R}. \tag{450}$$

Proof. For $0 \le r \le 1$, split

$$(\mathfrak{I}^{\beta}_{-,R} f)(r) = \frac{1}{\Gamma(\beta)} \int_r^1 (q-r)^{\beta-1} f(q)\, dq + \frac{1}{\Gamma(\beta)} \int_1^R (q-r)^{\beta-1} f(q)\, dq =: (\mathfrak{L}_\beta f)(r) + (\mathfrak{T}_{\beta,R} f)(r).$$

For the local part, extend $f|_{[0,1]}$ by zero to $\mathbb{R}$ and set

$$k_\beta(s) := \Gamma(\beta)^{-1} s^{\beta-1} \mathbf{1}_{(0,1)}(s).$$

Since $k_\beta \in L^1(\mathbb{R})$, Young's inequality gives

$$\|\mathfrak{L}_\beta f\|_E \le \|k_\beta\|_{L^1(\mathbb{R})} \|f\|_{L^2([0,1])} \le C_\beta \|f\|_{E_R}.$$

For the tail part, the kernel

$$K^{\mathrm{tail}}_{\beta,R}(r,q) := \Gamma(\beta)^{-1} (q-r)^{\beta-1} \mathbf{1}_{\{0 \le r \le 1,\ 1 \le q \le R\}}$$

is Hilbert–Schmidt uniformly in $R \ge 1$, because

$$\begin{aligned} \int_0^1 \int_1^R \left| K^{\mathrm{tail}}_{\beta,R}(r,q) \right|^2 dq\, dr &\le C_\beta \int_0^1 \int_1^\infty (q-r)^{2\beta-2}\, dq\, dr \\ &= \frac{C_\beta}{1-2\beta} \int_0^1 (1-r)^{2\beta-1}\, dr < \infty, \end{aligned}$$

since $\beta < \frac{1}{2}$. Hence

$$\|\mathfrak{T}_{\beta,R} f\|_E \le C_\beta \|f\|_{E_R}$$

with C_β independent of R. Combining the local and tail estimates proves (450). □

Lemma 33.49 (Head-window compatibility for cutoff-supported inputs)**.** *Assume the hypotheses of Theorem 19.4, let $R \ge 1$, and let $\chi \in C^\infty([0,1])$ with zero extension $\overline{\chi}$ to $[0,R]$. For*

$$\Phi \in H^\beta((0,R)), \qquad Z \in E_R,$$

set

$$\phi := \Phi|_{[0,1]}, \qquad z := Z|_{[0,1]}.$$

Then on $E = L^2([0,1])$,

$$\mathbf{1}_{[0,1]}\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}(\overline{\chi}Z) = \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}(\chi z), \tag{451}$$

$$\mathbf{1}_{[0,1]}\mathfrak{T}^{\mathrm{hl}}_{t,R}(\overline{\chi}\Phi) = \mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi), \tag{452}$$

$$\mathbf{1}_{[0,1]}\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,R}\mathfrak{T}^{\mathrm{hl}}_{t,R}(\overline{\chi}\Phi) = \mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi), \tag{453}$$

$$\mathbf{1}_{[0,1]}\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}(\overline{\chi}\Phi) = \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi). \tag{454}$$

Consequently,

$$\begin{aligned}\mathbf{1}_{[0,1]}\mathfrak{F}^{\mathrm{ex}}_{t,h,\alpha,R}(\overline{\chi}\,\Phi, h^{\alpha\beta}\overline{\chi}\,Z) = {}& \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}(\chi z)\\ &- L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}\mathfrak{T}^{\mathrm{hl}}_{t,1}(h^{\alpha\beta}\chi\phi)\\ &- \mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi).\end{aligned} \tag{455}$$

Proof. Since $\overline{\chi}Z$ is supported in $[0,1]$, the explicit Volterra formula from the proof of Proposition 15.3 shows that for $0 \le r \le 1$,

$$(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,R}(\overline{\chi}Z))(r)$$

depends only on the values of $\overline{\chi}Z$ on $[r,1]$, with the same kernel $L_Y(t-hr, t-hq)$ as in the window $R=1$. This proves (451).

Likewise, because $\overline{\chi}\Phi$ is supported in $[0,1]$, one has for $0 \le r \le 1$

$$\mathfrak{I}_{\beta,R}(\overline{\chi}\Phi)(r) = \int_r^R (q-r)^{-\beta}\overline{\chi}(q)\Phi(q)\,dq = \int_r^1 (q-r)^{-\beta}\chi(q)\phi(q)\,dq = \mathfrak{I}_{\beta,1}(\chi\phi)(r).$$

Taking the distributional derivative on $(0,1)$ and using Lemma 19.2(ii) gives (452).

For (453), note first that (452) identifies the two inputs on $[0,1]$. Moreover, the definition

$$m_{t,h,\alpha,R}(r) = \frac{g^{\mathrm{win}}_{t,h,R}(r,r)}{g^{\mathrm{hl}}_t}$$

and the formula defining $g^{\mathrm{win}}_{t,h,R}$ in Proposition 19.3 imply

$$g^{\mathrm{win}}_{t,h,R}(r,q) = g^{\mathrm{win}}_{t,h,1}(r,q) \qquad \text{for } 0 \le r \le q \le 1,$$

hence $m_{t,h,\alpha,R}(r) = m_{t,h,\alpha,1}(r)$ for $0 \le r \le 1$. This is (453).

For (454), use the explicit formula from Proposition 19.3. On $[0,1]$, both terms defining $\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,R}(\overline{\chi}\Phi)$ integrate only over $q \in [r,1]$, because $\overline{\chi}\Phi$ vanishes on $(1,R]$. Since

$$g^{\mathrm{win}}_{t,h,R} = g^{\mathrm{win}}_{t,h,1}, \qquad b_{t,h,\alpha,R} = b_{t,h,\alpha,1} \qquad \text{on } \Delta_1,$$

the two remainder formulas agree there, which proves (454).

Finally, (455) is the definition (225) combined with (451)–(454). □

Theorem 33.50 (Exact-to-frozen perturbation on normalized growing-window exact pairs)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then there exist $h_{0,\alpha} \in (0,t]$ and $C_{t,\alpha,\chi} < \infty$ such that for every*

$$0 < h \le h_{0,\alpha} \qquad \text{with} \qquad h^\alpha \le t,$$

and every $v \in \mathfrak{G}_t$,

$$\left\| \mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{Z}^{\mathrm{full}}_{v,h,\alpha}) - \mathfrak{E}^{\mathrm{fr}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{Z}^{\mathrm{full}}_{v,h,\alpha}) \right\|_E \le C_{t,\alpha,\chi} h \|v\|_{\mathfrak{G}_t}. \tag{456}$$

Proof. Shrink $h_{0,\alpha}$, depending only on t, α, χ, and the $R=1$ regularity thresholds entering Theorem 19.4, if necessary so that for every $0 < h \le h_{0,\alpha}$,

$$h \le 1, \qquad h^\alpha \le t, \qquad 1 < \varepsilon^{-1}_{h,\alpha},$$

and Theorem 19.4 applies on the fixed window $R=1$. Fix such an h, let

$$R := R_{h,\alpha}, \qquad \Phi := \widehat{\Phi}_{v,h,\alpha}, \qquad Z := \widehat{Z}^{\mathrm{full}}_{v,h,\alpha},$$

and write

$$\phi := \Phi|_{[0,1]}, \qquad z := Z|_{[0,1]}.$$

By Lemma 33.49, (225), (432), and (434) reduce on $[0,1]$ to

$$\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R,\chi}(\Phi,Z) = \mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1}(\chi z) - L_Y(t,t)\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1}h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi) - h^{-\alpha\beta}\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi), \tag{457}$$

$$\mathfrak{E}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z) = L_Y(t,t)\Big(\chi z - h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi)\Big). \tag{458}$$

Subtracting (458) from (457) gives

$$\begin{aligned}\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R,\chi}(\Phi,Z) - \mathfrak{E}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z) = {}& (\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1} - L_Y(t,t)I_E)(\chi z)\\ &- L_Y(t,t)(\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1} - I_E)h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi)\\ &- h^{-\alpha\beta}\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi).\end{aligned} \tag{459}$$

For the first term, (183) with $R=1$ gives

$$\Big\|(\mathfrak{B}^{\mathrm{mic}}_{Y,t,h,\alpha,1} - L_Y(t,t)I_E)(\chi z)\Big\|_E \le C_{t,\chi}h\|z\|_E. \tag{460}$$

For the second term, (458) implies

$$h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi) = \chi z - L_Y(t,t)^{-1}\mathfrak{E}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z),$$

so by (214) with $R=1$ and Corollary 33.47,

$$\begin{aligned}\Big\|L_Y(t,t)(\mathfrak{M}^{\mathrm{win}}_{t,h,\alpha,1} - I_E)h^{-\alpha\beta}\mathfrak{T}^{\mathrm{hl}}_{t,1}(\chi\phi)\Big\|_E &\le C_{t,\chi}h\left(\|z\|_E + \Big\|\mathfrak{E}^{\mathrm{fr}}_{t,h,\alpha,R,\chi}(\Phi,Z)\Big\|_E\right)\\ &\le C_{t,\alpha,\chi}h\|v\|_{\mathfrak{G}_t}.\end{aligned} \tag{461}$$

For the third term, Proposition 33.39 and Lemma 33.45(iv) give

$$\Phi = \mathbf{c}_t^{-1}h^{\alpha\beta}\mathfrak{I}^{\beta}_{-,R}Z \qquad \text{in } H^{\beta}((0,R)).$$

Hence

$$h^{-\alpha\beta}\chi\phi = \mathbf{c}_t^{-1}\chi\,\mathbf{1}_{[0,1]}\mathfrak{I}^{\beta}_{-,R}Z.$$

Using (215) with $R=1$ and Lemma 33.48,

$$\begin{aligned}\Big\|h^{-\alpha\beta}\mathfrak{K}^{\mathrm{win}}_{t,h,\alpha,1}(\chi\phi)\Big\|_E &\le C_t h\Big\|h^{-\alpha\beta}\chi\phi\Big\|_E\\ &\le C_{t,\chi}h\Big\|\mathbf{1}_{[0,1]}\mathfrak{I}^{\beta}_{-,R}Z\Big\|_E\\ &\le C_{t,\alpha,\chi}h\|Z\|_{E_R} \le C_{t,\alpha,\chi}h\|v\|_{\mathfrak{G}_t}.\end{aligned} \tag{462}$$

Finally, (431) gives

$$\|z\|_E \le \|Z\|_{E_R} \le \frac{\nu_Y}{\eta}\|v\|_{\mathfrak{G}_t}.$$

Combining this with (459), (460), (461), and (462) proves (456). □

Corollary 33.51 (Normalized growing-window endpoint bound)**.** *Assume the hypotheses of Proposition 33.5 and the extra regularity from Definition 19.1, and suppose moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then there exist $h_{0,\alpha} \in (0,t]$ and $C_{t,\alpha,\chi} < \infty$ such that for every*

$$0 < h \le h_{0,\alpha} \qquad \textit{with} \qquad h^{\alpha} \le t,$$

and every $v \in \mathfrak{G}_t$,

$$\Big\|\mathfrak{E}^{\mathrm{norm}}_{t,h,\alpha,R_{h,\alpha},\chi}(\widehat{\Phi}_{v,h,\alpha}, \widehat{Z}^{\mathrm{full}}_{v,h,\alpha})\Big\|_E \le C_{t,\alpha,\chi}\|v\|_{\mathfrak{G}_t}. \tag{463}$$

In particular, (433) *holds.*

Proof. Theorem 33.50 splits the normalized exact defect into the frozen commutator plus an error bounded by $C_{t,\alpha,\chi}h\|v\|_{\mathfrak{G}_t}$. Corollary 33.47 bounds the frozen commutator uniformly by $C_{t,\alpha,\chi}\|v\|_{\mathfrak{G}_t}$. After shrinking $h_{0,\alpha}$ if necessary, the two estimates give (463). The final assertion is the same bound written in the notation of (433). □

Theorem 33.52 (Closed source-screening under endpoint-flat cutoffs)**.** *Assume Assumptions 3.1 and 4.1, assume in addition the extra regularity from Definition 19.1, let $\eta > 0$, fix $t \in (0,T)$ and $\alpha \in (0,1)$, and suppose*

$$0 < H_{XY} < H_Y < \tfrac{1}{2}.$$

Choose

$$\chi \in C^\infty([0,1]), \qquad 0 \le \chi \le 1, \qquad \chi \equiv 1 \text{ on } [0, 1-\delta_*]$$

for some $\delta_* \in (0,1)$, and assume moreover that χ is flat at the outer edge in the sense of Definition 33.21. Then $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t, and

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = o(h^{H_{XY}}) \qquad \text{as } h \downarrow 0, \tag{464}$$

and consequently

$$\widetilde{G}_{X\to Y}(t,h) = o(h^{2H_{XY}}) \qquad \text{as } h \downarrow 0. \tag{465}$$

The theorem is closed in a specific sense. The cutoff stability required by Theorem 32.8 is verified by the endpoint-flat construction above. The statement is not cutoff-free; the endpoint-flat cutoff and the C^2 near-diagonal regularity remain part of the screening mechanism.

Proof. Corollary 33.51 gives the normalized growing-window defect bound (433). Proposition 33.41 converts that bound into the unnormalized estimate (426). Corollary 33.37 therefore supplies the endpoint estimate required by Corollary 33.31. Hence $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t. Applying Theorem 32.8 with these cutoffs gives (464); the gain estimate (465) then follows from the graph-dual identity (67). □

Corollary 33.53 (Endpoint-flat screening from unscaled exact correction)**.** *Assume the hypotheses of Theorem 33.7. Then*

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = o(h^{H_{XY}}) \qquad \text{as } h \downarrow 0, \tag{466}$$

and consequently

$$\widetilde{G}_{X\to Y}(t,h) = o(h^{2H_{XY}}) \qquad \text{as } h \downarrow 0. \tag{467}$$

Proof. Theorem 33.7 proves that $\mathcal{S}_t$ has h^α-stable boundary cutoffs at time t. Those cutoffs are the only additional hypothesis in Theorem 32.8. Substitution into that theorem gives (466), and the gain estimate follows from the graph-dual identity (67). □

34. Conclusion

Dynamic observability lies between threshold geometry and feasible inference. The threshold analysis of [1] separates pricing visibility from path-space detectability in the smoothing regime. Dynamic observability is the residual predictive quotient after latent drivers are replaced by observable histories.

The three levels have different boundaries. At finite resolution, the canonical Gaussian experiment generated by the relative covariance operator recovers

$$H_{XY} = H_Y + \tfrac{1}{4}$$

as the critical boundary for evidence accumulation with resolution. At the oracle latent-driver level, the threshold is instead

$$H_{XY} = H_Y,$$

the sharp criterion for leading-order short-horizon prediction gain. At the observable level, even that oracle gain can be screened. In the rougher regime $0 < H_{XY} < H_Y < \frac{1}{2}$, the screening theorem is first proved under an h^α-stable boundary-cutoff hypothesis. The exact-correction closure theorem verifies that hypothesis for the model classes treated here under the present second-order amplitude regularity assumptions. In the endpoint-flat closure theorem, the target asset's own past screens the dominant oracle contribution.

The conclusion is a three-level visibility structure for latent contagion. Pricing visibility, path-space detectability, and dynamic observability need not coincide. Volterra covariance operators govern the first comparison, Gaussian experiments govern the finite-resolution evidence scale, and the hidden transfer operator governs observable screening.

The screened observable component is the estimand for projected source-screened inference. The oracle object from the dynamic theorem is reduced by conditioning on the target channel and restricting information to observables; the remaining residual object is the inferential target. The construction identifies that target and its screening boundary.

The remaining problems are local to the assumptions and quantitative to the rate. The observable screening theorem uses C^2 amplitude regularity together with the cutoff-stability mechanism verified in the closure argument. The boundary of those assumptions is not settled. A quantitative refinement would start from Corollary 32.10. Once the near-layer profile ω_α admits a rate, the global observable screening bound inherits it. The open problem is to derive such a rate directly from the exact-correction closure for the model classes treated here. In particular, if the near-layer defect satisfies

$$\omega_\alpha(h) = O(h^\rho) \qquad \text{for some } \rho > 0,$$

then Corollary 32.10 yields the stronger screening estimate

$$\|\mathcal{F}_{t,h}\|_{\mathfrak{G}_t^*} = O(h^{H_{XY}+\varepsilon}),$$

with

$$\varepsilon = \min\{\rho,\ 1-\alpha(1-H_{XY}) - H_{XY},\ H_Y - H_{XY}\},$$

provided this minimum is positive. That quantitative rate problem belongs to finite-sample analysis and remains outside the present closure theorem. Extending the same observability problem to nonlinear, non-Gaussian, or jump-driven rough models would require tools beyond the Gaussian comparison and exact Volterra localization arguments used here.

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Email address: `vidal.llaurado@gmail.com`